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Annulets in a 0-distributive Lattice

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Abstract. The set of all ideals of the form $(x]^*$, $x \in L$ are known as annulets of L. These have been studied extensively by Cornish in [4, 5] for distributive lattices. In this paper we have studied the topic only for o-distributive lattices.

Keywords: 0-distributive lattice, Quassi-Complemented lattice, Disjunctive lattice, Annulets, S-algebra, DM algebra.

AMS Mathematics Subject Classification (2010): 06A12, 06A99, 06B10

1. Introduction

Varlet [8] has given the definition of a 0-distributive lattice to generalize the notion of pseudo complemented lattices. According to [8], a lattice L with 0 is called a 0-distributive lattice if for all $a,b,c \in L$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$. In other words, a lattice L with 0 is 0-distributive if and only if for each $a \in L$, the set of elements disjoint to a is an ideal of L. Of course, every distributive lattice with 0 is 0-distributive. Also every pseudo complemented lattice is 0-distributive. In fact, in a pseudo complemented lattice L, the set of all elements disjoint to $a \in L$, is a principal ideal $(a^*]$. Many authors including Balasubramani and Venkatanarasimhan [2], Jayaram [6] and Pawer and Thakare [7] studied the 0-distributive and 0-modular properties in different context and has included several characterizations to study a large class of non-distributive lattices with 0. Moreover, [9,10] studied on 0-distributive property for near lattices. Annulets in distributive lattices have been first studied by Cornish in [4, 5]. Then [1] extended the concept for Near lattices. In this paper our intension is study the annulets for a distributive lattice.

We know that if L is a 0-distributive lattice, then $I(L)$ is pseudo complemented. The set of all ideals of the form $(x]^*$; $x \in L$ are called the annulets of L.

For an ideal J of a lattice L we define $J^* = \{x \in L : x \wedge j = 0 \text{ for all } j \in J\}$. This is of course an ideal of L if L is 0-distributive and it is called the annihilator ideal. In fact when L is 0-distributive J^* is the pseudo complement of J in $I(L)$. Any ideal I is called an annihilator ideal if $I = I^{**}$. We denote the set of annihilator ideal of L by

$A(L)$. This can be made into a Boolean Algebra with smallest element (0], largest element L, set theoretical intersection as the infimum and the map $J \to J^*$ as complementation. The supremum of J and K in $A(L)$ is given by $J \underline{\vee} K = (J^* \cap K^*)^*$. This is no more than De-morgan's law.

Call an ideal of the form $(x]^*, x \in L,$ an annulet. Each annulet is an annihilator ideal and hence for two annulets $(x]^*$ and $(y]^*$, their supremum in $A(L)$ is $(x]^* \underline{\vee} (y]^* = ((x]^{**} \cap (y]^{**})^* = ((x \wedge y]^{**})^* = (x \wedge y]^*$. Also their infimum in $A(L)$ is $(x]^* \cap (y]^* = (x \vee y]^*$. The set of annulets is denoted by $A_0(L)$.

For a distributive lattice L, we know that $I(L)$ is pseudo complemented. So throughout the paper we always consider L as a 0-distributive lattice.

2. Annulets in a lattice

In this paper we have given several characterizations of Annulets for a 0-distributive lattice. We start the paper with following proposition:

Proposition 2.1. Let L be a 0-distributive lattice. Then $(A_0(L); \cap, \underline{\vee})$ is a lattice and a sublattice of the Boolean Algebra $(A(L); \cap, \underline{\vee}, *, (0], L)$ of Annihilator ideals of L. $A_0(L)$ has a smallest element if and only if L possesses an element $d \in L$ such that $(d]^* = (0]$.

Proof: We have already shown that $A_0(L)$ is a sublattice of $A(L)$. If there is an element $d \in L$ such that $(d]^* = (0]$. Then clearly $(0]$ is the smallest element in $A_0(L)$.

Conversely, if there is an element $d \in L$ such that $(d]^*$ is the smallest element then for any $x \in L, (x]^* = (x]^* \underline{\vee} (d]^* = (x \wedge d]^*$. Thus $x \wedge d = 0$ implies $(x]^* = (0]^* = L,$ so that $x = 0$ and hence $(d]^* = (0]$. ■

In a P-algebra $(L; \wedge, \vee, *, 0, 1)$ for all $a, b \in L$, $(a \vee b)^* = a^* \wedge b^*$ always holds, but the other De-Morgan identity $(a \wedge b)^* = a^* \vee b^*$ may not hold in general. Chandrani in [3] has called a P-algebra as a DM-algebra if $(a \wedge b)^* = a^* \vee b^*$ for all $a, b \in L$. A lattice L with 0 is called a generalized DM-algebra if for each $x, y \in L,$ $(x]^* \vee (y]^* = (x \wedge y]^*$.

Proposition 2.2. A lattice L with 0 is generalized DM-lattice if and only if $A_0(L)$ is a sublattice of the lattice of $I(L)$.

Proof: $A_0(L)$ is a sublattice of $I(L)$ if and only if for any $x, y \in L,$ $(x]^* \vee (y]^* = (z]^*$ for some $z \in L.$ Since $(x]^* \vee (y]^* = (z]^*$ implies $(z]^{**} = (x]^{**} \cap (y]^{**} = (x \wedge y]^{**}$, so that $(z]^* = (x \wedge y]^* = (x]^* \underline{\vee} (y]^*$ in $A_0(L)$, we see that $A_0(L)$ is a sublattice of $I(L)$ if and only if $(x]^* \vee (y]^* = (x \wedge y]^*$ for each $x, y \in L.$ By the definition of DM-algebra, this is equivalent to $L,$ being a DM-algebra. ■

A lattice L with 0 is called Disjunctive if for any $a, b \in L, a < b$ implies $a \wedge c = 0$ and $c < b$ for some $0 \neq c.$ However, it is easy to see that a lattice L with 0 is disjunctive if and only if $(a]^* = (b]^*$ implies $a = b$ for any $a, b \in L.$ We thus have the following corollary:

Corollary 2.3. A disjunctive lattice L with generalized De-Morgan property is dual isomorphic to its lattice of annulets. Hence L has a largest element if and only if there is an element $d \in L$ such that $(d]^* = (0].$ ■

A lattice L with 0 is called a quasi-complemented lattice if for each $x \in L,$ there exists $y \in L$ such that $x \wedge y = 0$ and $((x] \vee (y])^* = (x]^* \cap (y]^* = (0].$

A 0-distributive lattice L is called quasi-complemented if for each $x \in L,$ there exists $x' \in L$ such that $x \wedge x' = 0$ and $((x] \vee (x'])^* = (0].$

A lattice L with 0 is called sectionally quasi-complemented if each interval $[0, x],$ $x \in L$ is quasi-complemented.

Theorem 2.4. A 0-distributive lattice L is quasi-complemented if and only if for each $x \in L,$ there exists $y \in L$ such that $(x]^{**} = (y]^*.$

Proof: Let L be a quasi-complemented. Suppose $x \in L,$ there exists $y \in L$ such that $x \wedge y = 0$ and $(x]^* \cap (y]^* = (0].$ This implies $(y]^* \subseteq (x]^{**}.$

Again, $x \wedge y = 0$ implies $(x] \cap (y] = (0],$ so $(x] \subseteq (y]^*.$ Therefore, $(x]^{**} \subseteq (y]^{***} = (y]^*$ and hence $(x]^{**} = (y]^*.$

Conversely, let $x \in L$ implies $(x]^{**} = (y]^*$ for some $y \in L$. Then $x \in (x]^{**} = (y]^*$ implies $x \wedge y = 0$. Also $(x]^{**} = (y]^*$ implies
$(x]^* \cap (y]^* = (x]^* \cap (x]^{**} = (0],$ and so L is quasi complemented. ■

Theorem 2.5. Let L be a 0-distributive lattice. Then L is quasi-complemented if and only if it is sectionally quasi-complemented and possesses an element d such that $(d]^* = (0].$

Proof: Suppose L is quasi-complemented. Then there exists an element d such that $0 \wedge d = 0$ and $(d]^* = ((0] \vee (d])^* = (0].$ We now show that an arbitrary interval $[0, x]$ is quasi-complemented. Let $y \in [0, x].$ Then there exists $y' \in L$ such that $y \wedge y' = 0$ and $((y] \vee (y'])^* = (0].$ Put $z = x \wedge y'.$ Then $z \wedge y = (x \wedge y') \wedge y = x \wedge (y \wedge y') = 0$ and $z \in [0, x].$ If $w \in [0, x]$ and $(w] \wedge ((y] \vee (z]) = (0],$

then $(w \wedge y] = (0] = (w \wedge z] = (w \wedge x \wedge y'] = (w \wedge y']$. Thus $(w] \wedge ((y] \vee (y']) = (0]$ as L is 0-distributive and so $I(L)$ is 0-distributive. Hence $w = 0$, and so $[0, x]$ is quasi-complemented.

Conversely, suppose L is sectionally quasi-complemented and there exists an element $d \in L$ with $(d]^* = (0]$. Let $x \in L$ and consider the interval $[0, d]$. Then $x \wedge d \in [0, d]$.

Since L is sectionally quasi-complemented, so there exists an element $x' \in [0, d]$ with $x \wedge d \wedge x' = 0$ and $\{y \in [0, d] \mid y \wedge ((x \wedge d) \vee x' = 0\} = (0]$. Now let $z \in ((x] \vee (x'])^*$. Then $x \wedge r = 0$ for all $r \in (x] \vee (x']$.
Since $(x \wedge d) \vee x' \in (x] \vee (x']$, so $z \wedge ((x \wedge d) \vee x') = 0$. Thus

$$z \wedge d \wedge ((x \wedge d) \vee x') = 0 \text{ and } z \wedge d \in [0, d];$$

so $z \wedge d = 0$. This implies $z \in (d]^* = (0]$. Hence $z = 0$ and $x \wedge d \wedge x' = 0$ implies $x \wedge x' = 0$. Therefore L is quasi-complemented. ∎

Theorem 2.6. A 0-distributive lattice is quasi-complemented if and only if $A_0(L)$ is a Boolean subalgebra of $A(L)$.

Proof: Suppose L is quasi-complemented. Then by Theorem 2.5, L has an element d such that $(d]^* = (0]$. Then by proposition 2.1, $A_0(L)$ has a smallest element and so it is a sublattice of $A(L)$. Moreover, for each $x \in L$ there exists $x' \in L$ such that $x \wedge x' = 0$ and $(x]^* \cap (x']^* = (0]$. Then $(x]^* \underline{\vee} (x']^* = (x \wedge x']^* = (0]^* = L$. Therefore $A_0(L)$ is a Boolean subalgebra of $A(L)$.

Conversely, if $A_0(L)$ is a Boolean subalgebra of $A(L)$, then for any $x \in L$ there exists $y \in L$ such that $(x]^* \cap (y]^* = (0]$ and $(x]^* \underline{\vee} (y]^* = L$. But $(x]^* \underline{\vee} (y]^* = (x \wedge y]^*$ and $x \wedge y = 0$. Therefore, L is quasi-complemented. ∎

Let us introduce the following lemma, whose proof is trivial.

Lemma 2.7. Let $I = [0, x]$, $0 < x$ be an interval in a 0-distributive lattice. For $a \in I$, $(a]^+ = \{y \in I \mid y \wedge a = 0\}$ is the annihilator of $(a]$ with respect to I. Then

(i) If $a, b \in I$ and $(a]^+ \subseteq (b]^+$ then $(a]^* \subseteq (b]^*$

(ii) If $w \in L$, $(w]^* \cap I = (w \wedge x]^+$. ∎

The above lemma is useful to prove the generalization of the proposition 2.5 in [5] by Cornish. Let $I = [0, x]$, $0 < x$ be an interval in a distributive lattice with 0. For $a \in I$, $(a]^+ = \{y \in I \mid y \wedge a = 0\}$ is the annihilator of $(a]$ with respect to I. Then

(i) If $a, b \in I$ and $(a]^+ \subseteq (b]^+$ then $(a]^* \subseteq (b]^*$

(ii) If $w \in L$, $(w]^* \cap I = (w \wedge x]^+$.

Theorem 2.8. For a 0-distributive lattice L, $A_0(L)$ is relatively complemented if and only if L is sectionally quasi-complemented.

Proof: Suppose $A_0(L)$ is relatively complemented. Consider the interval $I=[0,x]$ and let $a\in I$; then $(x]^*\subseteq(a]^*\subseteq(0]^*=L.$ Since $[(x]^*,L]$ is complemented in $A_0(L)$, there exists $w\in L$ such that $(a]^*\cap(w]^*=(x]^*$ and $(a]^*\underline{\vee}(w]^*=L.$ Then $(a]^*\underline{\vee}(w]^*=(a\wedge w]^*$ gives $a\wedge w=0$. Then $a\wedge w\wedge x=0$ and $w\wedge x\in I.$ Moreover, intersecting $(a]^*\cap(w]^*=(x]^*$ with $(x]$ and using the Lemma 2.7, we have $(a]^+\cap(w\wedge x]^+=(0]$.

This shows that *I* is quasi-complemented.

Conversely, suppose L is sectionally quasi-complemented. Since $A_0(L)$ is 0-distributive, it suffices to prove that the interval $[(a]^*,L]$ is complemented for each $a\in L.$ Let $(b]^*\in[(a]^*,L].$ Then $(a]^*\subseteq(b]^*\subseteq L,$ so $(b]^*=(a]^*\underline{\vee}(b]^*=(a\wedge b]^*.$ Now consider the interval $I=[0,a]$ in $L.$ Then $a\wedge b\in I.$ Since I is quasi-complemented, there exists $w\in I$ such that $w\wedge a\wedge b=0$ and $(w]^+\cap(a\wedge b]^+=(0]=(a]^+.$ This implies $(w\vee(a\wedge b)]^+=(a]^+.$ Then $(a]^*=(w\vee(a\wedge b)]^*=(w]^*\cap(a\wedge b]^*=(w]^*\cap(b]^*.$ also $w\wedge a\wedge b=0$ we have $w\wedge b=0,$ hence $(w]^*\underline{\vee}(b]^*=L.$ Therefore, $A_0(L)$ is relatively complemented. ■

By Chandrani in [3], a P-algebra $(L;\wedge,\vee,*,0,1)$ is called a S-algebra if for each $a\in L$, $a^*\vee a^{**}=1.$ Thus, a 0-distributive lattice L is called a generalized S-lattice if for each $x\in L,$ $x^*\vee x^{**}=L.$ By [2], every DM-algebra is an S-algebra but the converse need not be true.

We conclude the paper with the following result:

Proposition 2.9. The lattice of annulets of a lattice *L* with 0 is relatively complemented if and only if *L* is quasi-complemented.

Proof: Suppose $A_0(L)$ is relatively complemented. We must show that $I=[0,x]$ is a quasi-complemented lattice for each $0<x\in L.$ Let $a,b\in I$ and suppose $(a]^+\subseteq(b]^+\subseteq I=(0]^+.$ From the lemma, $(a]^*\subseteq(b]^*\subseteq L.$ The interval $[(a]^*,L]$ is complemented in $A_0(L)$ so that there is an element $w\in L$ such that $(b]^*\cap(w]^*=(a]^*$ and $(w]^*\underline{\vee}(b]^*=L.$ Then $(b]^*\underline{\vee}(w]^*=(b\wedge w]^*$ gives $b\wedge w=0.$

Then $b\wedge(w\wedge x)=0$ so $(b]^+\underline{\vee}(w\wedge x]^*=(a]^*$ due to lemma 2.7. It follows that $A_0(L)$ is complemented and so by proposition 2.6, I is quasi-complemented.

Suppose L is sectionally quasi-complemented. To prove $A_0(L)$ is relatively complemented it suffices to prove that each interval $[(a]^*,L]$ is complemented as $A_0(L)$ is distributive (Proposition 2.1). Let $(b]^*\in[(a]^*,L]\subseteq A_0(L)$ and consider the interval

$I=[0,a\vee b]$ in L. Then $(a]^{+}=(a]^{*}\cap I\subseteq(b]^{*}\cap I=(b]^{+}\subseteq I$ so there is an element $w\in I$ such that $(b]^{+}\cap(w]^{+}=(a]^{+}$ and $(w]^{+}\underline{\vee}(b]^{+}=I$ as I is quasi-complemented and so $A_0(L)$ is complemented by Proposition 2.7. Then $(w\vee b]^{+}=(w]^{+}\cap(b]^{+}=(a]^{+}$. ■

3. Conclusion

Cornish [5] has studied Annulets and $\alpha-$ideals in a distributive lattice. In order to study these for non distributive lattices we studied to apply these results in *0*-distributive lattice. Recently, Ayub and Podder have introduced the concept of n-distributive lattices where *n* is a central element. Of course the set of all principal n-ideals of a lattice is again a lattice when *n* is a central element. Therefore, our specific suggestion is one can extend the results of this paper for $\mathrm{P}_n(L)$.

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