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On Solutions to the Diophantine Equation $M^x + (M + 6)^y = z^2$ when $M = 6N + 5$

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Abstract. In this article we investigate solutions to the title equation. We establish: (i) For all values M and even values x, y, then the equation has no solutions. (ii) When M, $M + 6$ are primes, and x, y interchange odd and even values, then the equation has a unique solution. (iii) If M is prime or composite and so is $M + 6$, then when $x = y = 1$ the equation has infinitely many solutions. In this case, a sufficient condition for a solution is determined. For all values $M < 200$ and $x = y = 3$, then the equation has no solutions.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-liniar such individual equations involving primes, composites and powers of all kinds. Among them are [2, 3, 8, 9, 10].

A prime gap is the difference between two consecutive primes. Articles as [4, 6] and many others have been written on prime gaps. In 1849, A. de Polignac conjectured that for every positive integer k, there are infinitely many primes p such that $p + 2k$ is prime too. Many questions and conjectures on the above still remain unanswered and unsolved.

When $k = 1$, the pairs $(p, p + 2)$ are known as Twin Primes. The first four such pairs are: (3, 5), (5, 7), (11, 13), (17, 19). The Twin Prime conjecture stating that there are infinitely many such pairs remains unproved. When $k = 2$, the pairs $(p, p + 4)$ are called Cousin Primes. The first four pairs are: (3, 7), (7, 11), (13, 17), (19, 23). The conjecture that there are infinitely many Cousin Primes is still unsettled.

In this article, we concern ourselves with the case $k = 3$, i.e., pairs of primes of the form $(p,\ p + 6)$. These pairs are named in the literature as "Sexy Primes" since "*sex*" in *Latin* means "six". The first four such pairs are: (5, 11), (7, 13), (11, 17), (13, 19). As of today, it is not known whether or not there exist infinitely many Sexy pairs.

The authors in [5] and [7] concern themselves with $p^x + (p + 6)^y = z^2$ where p, $(p+6)$ are primes and $p = 6N + 1$. It is shown [5] that the equation has no solutions, whereas in [7] particular cases of the equation are considered. The author [1] establishes certain results and solutions of this equation when p, $(p + 6)$ are primes, $p = 6N + 5$ and $x + y = 2, 3, 4$.

In this article, we investigate solutions of

$$M^x + (M + 6)^y = z^2 \qquad (1)$$

when $M = 6N + 5$. If M, $M + 6$ are primes, x, y are even or if one of them is, then equation (1) has exactly one solution. If M is prime or composite and so is $M + 6$, then for all such cases with $x = y = 1$, equation (1) has infinitely many solutions. A solution with primes M, $M + 6$ when $x = 5$ and $y = 1$ is also exhibited.

This is done in a series of self-contained theorems and tables.

2. Solutions of $M^x + (M + 6)^y = z^2$

Theorem 2.1. Let $N \geq 0$, $n \geq 1$, $m \geq 1$ be integers. If $M = 6N + 5$, then for all values M, n, m, the equation

$$M^{2n} + (M + 6)^{2m} = z^2 \qquad (2)$$

has no solutions.

Proof: For all values M, n, m, one can easily see that M^{2n} and $(M + 6)^{2m}$ are of the form $4A + 1$ and $4B + 1$. If (2) exists, then z^2 is even, and hence $z = 2T$. From (2) we then obtain

$$(4A + 1) + (4B + 1) = 2(2A + 2B + 1) = 4T^2$$

which is impossible.

Equation (2) has no solutions as asserted. □

Remark 2.1. In Theorem 2.1, it was shown that (2) has no solutions for all values M. Hence, the result is in particular true when M and $M + 6$ are primes.

Hereafter, when M and $M + 6$ are primes, we shall use the notation $M = p$ and $M + 6 = P$.

We now consider equation (1) i.e., $M^x + (M + 6)^y = z^2$ with $M, (M + 6)$ as primes p and P. In Theorem 2.2 we consider the case when x is even and y is odd, whereas Theorem 2.3 deals with odd x and even y.

Theorem 2.2. Let $n \geq 1$, $t \geq 0$ be integers. If $M = p$ and $M + 6 = P$, then for all values p, P, n, t, the equation

$$p^{2n} + P^{2t+1} = z^2 \tag{3}$$

has a unique solution when $p = 5$, $P = 11$, $n = 1$, $t = 0$, and $z = 6$.

Proof: From (3) we have

$$P^{2t+1} = z^2 - p^{2n} = z^2 - (p^n)^2 = (z - p^n)(z + p^n). \tag{4}$$

Since P is prime, the $(2t + 2)$ divisors of P^{2t+1} are 1, P^1, P^2, … , P^{2t}, P^{2t+1}. Thus, from (4) the first $(t + 1)$ possibilities are:
$z - p^n = 1$ and $z + p^n = P^{2t+1}$, $z - p^n = P^1$ and $z + p^n = P^{2t}$, … , $z - p^n = P^t$ and $z + p^n = P^{t+1}$.
Observe that the last $(t + 1)$ possibilities are a priori eliminated.

We now examine the first possibility.
Suppose that $z - p^n = 1$ and $z + p^n = P^{2t+1}$.
The two equalities imply

$$2p^n = P^{2t+1} - 1. \tag{5}$$

We will show that (5) yields exactly one solution.
In (5) when $n = 1$ and $t \geq 1$, we have

$$2p^1 + 1 = P^{2t+1} = (p + 6)^{2t+1}$$

which does not exist.
Moreover from (5), when $n = 1$ and $t = 0$, then

$$2p^1 = P^1 - 1$$

or $2p = (p + 6) - 1 = p + 5$. Hence, $p = 5$ and $P = 11$ implying a solution of (3), namely

Solution 1. $$5^2 + 11^1 = 6^2.$$

Rewriting (5), i.e., $2p^n = P^{2t+1} - 1^{2t+1}$, results in

$$p^n = \big((P - 1)/2\big)\big(P^{2t} + P^{2t-1} + \cdots + P^2 + P^1 + 1\big) \tag{6}$$

where $(P - 1)/2 > 1$ is an integer. Since p, P are primes, it then follows for all values $n > 1$ and $t \geq 0$ that (6) is impossible.

The first possibility is complete.

We now consider all the remaining t possibilities for $z - p^n$.
Suppose that $z - p^n = P^u$ where $1 \leq u \leq t$, and $z + p^n = P^{(2t+1)-u}$. Hence, $z = p^n + P^u$ yields

$$2p^n + P^u = P^{(2t+1)-u}. \tag{7}$$

For all values $n \geq 1$ and $t \geq 1$, the two sides of (7) are clearly in contradiction since p, P are primes. Therefore (7) is impossible. The equation $p^{2n} + P^{2t+1} = z^2$ has no solutions when $n \geq 1$ and $t \geq 1$.

Thus, when $n = 1$ and $t = 0$, **Solution 1** is the unique solution of the equation as asserted.

This concludes the proof of **Theorem 2.2**. □

Theorem 2.3. Let $n \geq 0$, $t \geq 1$ be integers. If $M = p$ and $M + 6 = P$, then for all values p, P, n, t, the equation

$$p^{2n+1} + P^{2t} = z^2 \qquad (8)$$

has no solutions.

Proof: From (8) we obtain

$$p^{2n+1} = z^2 - P^{2t} = z^2 - (P^t)^2 = (z - P^t)(z + P^t). \qquad (9)$$

Since p is prime, the $(2n + 2)$ divisors of p^{2n+1} are 1, p^1, p^2, … , p^{2n+1}. Thus, from (9) the first $(n + 1)$ possibilities are:
$z - P^t = 1$ and $z + P^t = p^{2n+1}$, $z - P^t = p^1$ and $z + P^t = p^{2n}$, … , $z - P^t = p^n$ and $z + P^t = p^{n+1}$, where the last $(n + 1)$ possibilities are a priori eliminated.

We now consider the first possibility.
Suppose that $z - P^t = 1$ and $z + P^t = p^{2n+1}$. The two equalities imply

$$2P^t = p^{2n+1} - 1. \qquad (10)$$

When $n = 0$, we obtain $2P^t = p^1 - 1$ which is impossible for all values $t \geq 1$ since $P > p$.

Rewriting (10), we have that $2P^t = p^{2n+1} - 1^{2n+1}$ yields

$$P^t = \big((p - 1)/2\big)\big(p^{2n} + p^{2n-1} + \cdots + p^2 + p^1 + 1\big) \qquad (11)$$

where $1 < (p - 1)/2 < P$ is an integer. For all values $n > 0$, $t \geq 1$ and since P is prime, it follows that (11) does not exist. Therefore, for all $n \geq 0$ and $t \geq 1$, equation (8) has no solutions.

This concludes the first possibility.

We shall now examine the remaining n possibilities for $z - P^t$.
Suppose that $z - P^t = p^v$ where $1 \leq v \leq n$, and $z + P^t = p^{(2n+1)-v}$. Thus, $z = P^t + p^v$ yields

$$2P^t + p^v = p^{(2n+1)-v}. \qquad (12)$$

Since p, P are primes, then for each value $n \geq 1$ and $t \geq 1$, the two sides of (12) are contradictory. Thus (12) is impossible.
When $n \geq 0$ and $t \geq 1$, the equation $p^{2n+1} + P^{2t} = z^2$ has no solutions.

The proof of **Theorem 2.3** is complete. □

The remaining part of this article is concerned with solutions of $M^x + (M + 6)^y = z^2$ when x, y are odd. The general case presents great difficulties, and we shall consider only two cases, namely: the case $x = y = 1$, and the case $x = y = 3$. This is done in Theorem 2.4.

Theorem 2.4. Let $N \geq 0$, $n \geq 0$, $t \geq 0$ be integers. If $M = 6N + 5$, then the equation

$$M^{2n+1} + (M + 6)^{2t+1} = z^2 \tag{13}$$

has:

(a) Infinitely many solutions when $n = t = 0$.
(b) No solutions for all values $M < 200$ when $n = t = 1$.

Proof: **(a)** Suppose that $n = t = 0$ in (13).

To begin with, we remark that we do not intend to find all the solutions of (13) when $n = t = 0$, but rather show the existence of infinitely many solutions in this case.

When M and $M + 6$ are composites, we shall hereafter use the notation $M = c$ and $M + 6 = C$.

In the following **Table 1**, we exhibit nine solutions of (13) in which $n = t = 0$. These solutions comprise the four existing types of solutions, namely:

$$(M,\ M + 6) = (p,\ P),\quad (c,\ P),\quad (p,\ C),\quad (c,\ C).$$

The solutions appear in this order.

Table 1. Solutions of $M^x + (M + 6)^y = z^2$ when $x = y = 1$.

Solution	$M = p$	$M = c$	$M + 6 = P$	$M + 6 = C$	z^2
Solution 1	5		11		4^2
Solution 2	47		53		10^2
Solution 3	6047		6053		110^2
Solution 4		95	101		14^2
Solution 5		125	131		16^2
Solution 6	29			35	8^2
Solution 7	797			803	40^2
Solution 8		12797		12803	160^2
Solution 9		39197		39203	280^2

Each type described in **Table 1** occurs infinitely many times.

Observe first that there exist infinitely many primes/composites in each of the following two progressions:

$8K - 1$: 7, 15, 23, 31, 39, 47, … , 95, … ,
and
$8K + 5$: 13, 21, 29, 37, 45, 53, … , 101, … .

Each three columns in both progressions, respectively represent integers of the form $6L+1$, $6L+3$, $6L+5$.
Since $M = 6N + 5$, a solution of (13) is obtained when

$$M^1 + (M + 6)^1 = 12N + 16 = 4(3N + 4) = z^2$$

implying that z^2 is even, and denote $z = 2T$. Thus $3N + 4 = T^2$, and T is odd or T is even. To prove our assertion, it suffices to consider anyone of the two possibilities. Suppose that T is odd. Denote $T = 2R + 1$, hence $3N + 4 = (2R + 1)^2$ or $3(N + 1) = 4R(R + 1)$. Then, $3 \mid R$ or $3 \mid (R + 1)$. It suffices to assume only $3 \mid R$. Denote $R = 3S$ where $S \geq 1$ is an integer. Thus, $N = 4S(3S + 1) - 1$. We then obtain

$$\begin{cases} M = 6N + 5 = 6(4S(3S + 1) - 1) + 5 = 8\big(3S(3S + 1)\big) - 1 = 8K - 1, \\ M + 6 = 6N + 11 = 6(4S(3S + 1) - 1) + 11 = 8(3S(3S + 1)) + 5 = 8K + 5, \end{cases} \tag{14}$$

where K is the product of two consecutive integers $(3S)$ and $(3S + 1)$.

Finally,

$$z^2 = 4(3N + 4) = 4(2R + 1)^2 = 4(6S + 1)^2. \tag{15}$$

In (14) and (15), infinitely many integers $S = 1, 2, 3, \dots, k, \dots$ yield infinitely many values M, $M + 6$, z^2, namely

$$M = 8\big(3S(3S + 1)\big) - 1, \quad M + 6 = 8\big(3S(3S + 1)\big) + 5, \quad z^2 = 4(6S + 1)^2$$

which satisfy the identity

$$M^1 = (M + 6)^1 = z^2.$$

Hence, each and every integer S determines a solution of the identity.

Thus, equalities (14) and (15) establish a sufficient condition for an infinitude of solutions to equation (13) when $n = t = 0$.

Part **(a)** is complete.

For the convenience of the reader, **Table 2** demonstrates nine solutions of $M^1 + (M + 6)^1 = z^2$ when $1 \leq S \leq 9$.

Table 2. Solutions of $M^1 + (M + 6)^1 = z^2$ when $1 \leq S \leq 9$.

Solution	***S***	***M***	***M* + 6**	z^2	**Type of Solution**
Solution 1	1	95	101	14^2	(c, P)
Solution 2	2	335	341	26^2	(c, C)
Solution 3	3	719	725	38^2	(p, C)
Solution 4	4	1247	1253	50^2	(c, C)
Solution 5	5	1919	1925	62^2	(c, C)
Solution 6	6	2735	2741	74^2	(c, P)
Solution 7	7	3695	3701	86^2	(c, P)
Solution 8	8	4799	4805	98^2	(p, C)
Solution 9	9	6047	6053	110^2	(p, P)

Remark 2.2. All four types of solutions are represented in **Table 2**. It is noted that **Solutions 1** and **9** here coincide respectively with **Solutions 4** and **3** in **Table 1**.

Remark 2.3. It is shown in [1] for the first 10000 primes when p, $(p + 6)$ are primes and $x = y = 1$, that the equation $p^x + (p + 6)^y = z^2$ has exactly seven solutions (type (p, P)), all of which are exhibited. Three of these solutions are **Solutions 1 – 3** in **Table 1**, the other four solutions are not demonstrated here.

(b) Suppose that $n = t = 1$ in (13).

There are 33 values M when $M < 200$. The 33 values $5 \le M \le 197$ have been examined in (13) when $n = t = 1$, and $M^3 + (M + 6)^3 = z^2$ has no solutions.

This concludes part **(b)**, and **Theorem 2.4**. □

Enlarging the value M in **(b)** requires the aid of a computer.

Remark 2.4. In [1] it is established: If $n = 0$, $t = 1$, and if $n = 1$, $t = 0$, then for all primes p, $(p + 6)$, the equation $p^{2n+1} + (p + 6)^{2t+1} = z^2$ has no solutions.

3. Conclusion

The odd prime $p = 5$ is a unique one. No other prime has a last digit which is equal to 5. It is quite evident that the values $M = p = 5$ and $M + 6 = p + 6 = 11$ have a particular role in the equation $M^x + (M + 6)^y = z^2$. First, we have the unique solution

Solution 1. $5^2 + 11^1 = 6^2$ $(x = 2, \quad y = 1)$.

Secondly, in **Solution 1** of **Table 1**, we have

$$5^1 + 11^1 = 4^2 \qquad (x = 1, \quad y = 1).$$

Finally, another solution is given by

Solution 2. $5^5 + 11^1 = 56^2$ $(x = 5, \quad y = 1)$.

Observing that $5^3 + 11^1 \ne z^2$, $5^7 + 11^1 \ne z^2$, $5^9 + 11^1 \ne z^2$, $5^{11} + 11^1 \ne z^2$. Further calculations require a computer.

Consider equation (13) i.e., $M^{2n+1} + (M + 6)^{2t+1} = z^2$ when $M = 5$. **Solution 1** of **Table 1** and **Solution 2** are solutions of this equation. The following question may now be raised.

Question 1. Does the equation $5^{2n+1} + 11^{2t+1} = z^2$ have any other solutions ?

If the answer is indeed negative to **Question 1**, then **Solution 2** with $n > 0$ is therefore unique. Moreover, when $M = 5$, then $5^x + 11^y = z^2$ has exactly three solutions in all of which $y = 1$ as shown above.

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