Annals of Pure and Applied Mathematics
Vol. 18, No. 2, 2018, 147-173
ISSN: 2279-087X (P), 2279-0888(online)
Published on 3 November 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v18n2a4

Annals of Pure and Applied Mathematics

On the Duhamel's Solutions to the Null Equations of Incompressible Fluids

Javier Rivera

865 River Boat Circle, Orlando FL 32828
Email: caribe31415@gmail.com

Received 10 October 2018; accepted 1 November 2018

Abstract. The incompressible Navier-Stokes equation with null initial conditions or simply null Navier-Stokes equation was developed from the incompressible Navier-Stokes equations by subtracting the incompressible Navier-Stokes equations evaluated at the initial time, 0, from itself at some future time, t. A solution of the null Navier-Stokes is obtained via Laplace transform valid for a finite time interval to obtain Duhamel's solution. Additionally, by setting the kinematic viscosity to zero the solution becomes a solution for the incompressible null Euler Equations for all t in $[0, \infty)$. Theorem 1 and Lemma 1 shows the methodology to prove Duhamel's solution satisfies both the divergence equation for incompressible fluids and the incompressible null Navier-Stokes momentum equations. Theorem 2 shows how to obtain the incompressible Euler solution by taking the limit of the kinematic viscosity to zero on the Duhamel's Navier Stokes solution vector function. This article demonstrates a clear path of how the solutions for the null Navier-Stokes and null Euler equations are obtained.

Keywords: Duhamel's solutions, Incompressible Navier-Stokes equations, Incompressible Euler equations

AMS Mathematics Subject Classification (2010): 76A02, 35Q30, 76D05, 76N10

1. Introduction

This article was inspired by Duhamel's formulas in equations 19 [3] and 1.5 [4] in Professor Terrence Tao impressive articles. This article develops the concepts and methods of solving the incompressible null Navier-Stokes and the incompressible null Euler equations via the Laplace transform and the Newtonian potential found in [3]. The Eulerian velocity, $u_k(t, X_k(t))$, is usually prescribed in the academic literature [1] as $u_k(t, x_k)$were the spatial Eulerian coordinate is a normal size letters, and spatial time coordinates, x_k, not an explicit function of time (i.e. no time arguments). In this article, the approach is different, we treat both Eulerian (Caps Letters) and Lagrangian (normal size letters) spatial coordinates for the same spatial location and time to represent the same fluid parcel, and since there is no difference between them, therefore both Eulerian and Lagrangian velocities are identical for the same location and time as found in the sampling process of reference [6] but if the times are different they may represent different fluid parcels which may happen to pass through the same location. Therefore, the Eulerian velocities field with Eulerian spatial coordinates are denoted as$u_k(t, X_k(t))$. In this article the Eulerian spatial coordinate, $X_k(t)$, is replaced by the Lagrangian spatial coordinate, i.e.

$u_k(t, X_k(t))|_{X_k(t)=x_k(t,x_{ok})} = u_k\big(t, x_k(t, x_{ok})\big)$ where the Lagrangian spatial coordinate argument, $x_k(t, x_{ok})$ is a function of time and has a parameter argument initial location, x_{ok},which is not a function of time and it represents where the fluid parcel crossed some stream line at initial coordinate x_{ok}. Therefore, if Lagrangian and Eulerian spatial coordinate locations are identical,$x_k(t, x_{ok}) = X_k(t)$at a given time t (See reference [6] Section 3), then they maybe not at any other time for that fluid parcel, i.e. typically$x_k(t, x_{ok}) \neq x_k(\tau, x_{ok})$, unless of course,$t = \tau$, or a periodic process is occurring. Partial time differentiation of $x_k(t, x_{ok}) = X_k(t)$ yields

$$u_k^{Lagrangian}\big(t, x_k(t, x_{ok})\big) = \frac{\partial\, x_k(t, x_{ok})}{\partial t} = \frac{\partial X_k(t)}{\partial t} = \frac{dX_k(t)}{dt} = u_k^{Eulerian}(t, X_k(t))$$

Both velocity representations are numerically equal. Therefore, the spatial representation used in this article is Lagrangian, while the flow field representation is Eulerian[1]. Following reference [6] each time particular time can be treated as a sampling point along all the streamlines with all the path-lines, which may be composed, of extremely large number of different fluid parcels path-lines crossing points in the streamline as shown by the sampling process in [6], although the notation used in [6] was a vector field notation (upper arrows) and the notation used here is component index notation. In this article we treat every point as a possible intersection of streamlines and path-lines in the flow field, not just a sampling of a single streamline as in [6].

The Navier-Stokes and Euler equations are nonlinear equations, which no exact solution has been found so far for the most general cases. For every time t $\geq$ 0, the divergence of the incompressible fluid flow is given by (Eq. 1)[2].

$$\sum_k \frac{\partial u_k\big(t, x_k(t, x_{ok})\big)}{\partial X_k} = 0 \tag{1}$$

The incompressible Navier-Stokes momentum equations are given by (Eq. 2).

$$\frac{\partial u_k\big(t, x_k(t, x_{ok})\big)}{\partial t} + \sum_j u_j\big(t, x_k(t, x_{ok})\big)\frac{\partial u_k\big(t, x_k(t, x_{ok})\big)}{\partial X_j} - \nu\Delta u_k\big(t, x_k(t, x_{ok})\big) = -\Big(\frac{\partial \phi(x_k(t, x_{ok}))}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p(x_k(t, x_{ok}))}{\partial X_k}\Big) \tag{2}$$

The field or material derivative is given as

$$\frac{du_k\big(t, x_k(t, x_{ok})\big)}{dt} = \frac{\partial u_k\big(t, x_k(t, x_{ok})\big)}{\partial t} + \sum_j u_j\big(t, x_k(t, x_{ok})\big)\frac{\partial u_k\big(t, x_k(t, x_{ok})\big)}{\partial X_j}.$$

where $\frac{d}{dt}$ is the material or field derivative [6], ϕ is the external force potential, p is the pressure, ρ_o is the constant density, and ν is the constant kinematic viscosity (in meter squared per second). Moving the Laplacian term to the left side of the equation to obtain (Eq. 3).

$$\Big(\frac{d}{dt} - \nu\Delta\Big) u_k\big(t, x_k(t, x_{ok})\big) = -\Big(\frac{\partial \phi(x_k(t, x_{ok}))}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p(x_k(t, x_{ok}))}{\partial X_k}\Big) = -\frac{\partial}{\partial X_k}\Big(\phi + \frac{p}{\rho_o}\Big) \tag{3}$$

Notice that when t = 0, in (Eq. 2) this equation becomes,

---

[1]Note: At this point Eulerian and Lagrangian field velocities are shown to be identical, therefore we use Eulerian field velocities nomenclature to follow the historical nomenclature. This statement is a repetition of a statement in Section 3.3 of reference [6].

[2]Note: The fluid Eulerian velocity is denoted by normal size letter *u*, except for the fluid velocity with null initial conditions, which is denoted by capital letter *U*. In this article, all fluid velocities are Eulerian, but they have Lagrangian spatial coordinates. The Einstein notation convention is not used in this article. Only variables with indices in the explicit sigma symbol are being sum, the indices always are equal to 1, 2 and 3 but not shown.

$$\frac{\partial u_{oi}}{\partial t} + \sum_j u_{oj} \frac{\partial u_{oi}}{\partial X_j} - \nu \Delta u_{oi} = -(\frac{\partial \phi_o}{\partial X_i} + \frac{1}{\rho_o} \frac{\partial p_o}{\partial X_i})$$

With initial conditions along the streamlines of the velocity field are given by (Eq. 4).

$$u_k(0, x_{ok}) = u_{ok} \quad (4)$$

The Euler momentum equation is obtained by setting the kinematic viscosity to zero in (Eqs. 2 or 3) to obtain (Eq. 5).

$$\frac{du_k(t, x_k(t, x_{ok}))}{dt} = \frac{\partial u_k}{\partial t} + \sum_j u_j \frac{\partial u_k}{\partial X_j} = -\frac{\partial}{\partial X_k}\left(\phi + \frac{p}{\rho_o}\right) \quad (5)$$

The above equations are valid throughout the fluid, but we will concentrate on the movement or flow of a parcel of fluid moving through or crossing a streamline and the center of the fluid parcel volume, *V(t)*, is $\boldsymbol{x_k(t, x_{ok})}$.

In Section 2, the null Navier-Stokes equations was developed from the incompressible Navier-Stokes equations by subtracting the incompressible Navier-Stokes equations evaluated at the initial time, 0, from itself at some future time, t.

In Section 3 consists of finding the solutions to the null Navier-Stokes equations via Laplace transform. Although we will check the Duhamel's solution satisfy the null Navier-Stokes equations with the understanding the time dependent nonlinear terms are nulled out. Section 3 contains two theorems and a lemma, which proves the Duhamel's solutions do indeed, solves both Navier-Stokes equations and Euler equations with the understanding the time dependent nonlinear terms are nulled out for incompressible fluids, although, at the expense of practical applications. This article shows a clear path of how the solution is obtained, but the Duhamel's solution does not contribute to a solution which includes the nonlinear part of a field derivative, since it actually zeros or nulls out the nonlinear time dependent terms (see Appendix A). The Duhamel's solution, which solves the null Navier Stoke equations, was found via Laplace transforms to be a convolution integral as shown by (Eq. 6)

$$U_k(t, x_k(t, x_{ok})) = u_k(t, x_k(t, x_{ok})) - u_{ok} = -\int_0^t (e^{(t-\tau)\nu\Delta} - 1) \frac{\partial W_k(\tau, x_k(\tau, x_{ok}))}{\partial \tau} d\tau \quad (6)$$

where, Δ is the Laplacian operator with respect to the spatial Eulerian coordinates,X_k,(i.e. $\Delta \equiv \Delta_{\vec{X}}$)of a fluid parcel. Both symbols Δ and $\Delta_{\vec{X}}$are used interchangeably. The Lagrangian coordinate $x_k(t, x_{ok})$in the argument of the Eulerian fluid velocity,$u_k(t, x_k(t, x_{ok}))$,is the center coordinates of the fluid parcel at time t with spherical control volume, $V(t)$,where as $x_k(\tau, x_{ok})$ in $W_k(\tau, x_k(\tau, x_{ok}))$ is the Lagrangian spatial coordinate center of other fluid parcel at other previous time τin the spherical control volume, $V(\tau)$. Both may have different spatial locations and different times; therefore, they **may** represent different parcels of fluids even though they may have started from the same location, x_{ok}. This description of x_{ok} can be visualized as a movement of an inserted tiny drop at $x_{ok}$of colored fluid with the same density as the rest of the fluid, at first the tiny drop is concentrated in a very small space and then it moves with the currents as time passes. The kernel,$K(t, \tau) = (e^{(t-\tau)\nu\Delta} - 1)$, is the kernel operator which is also a transfer function of time, and the Laplacian operator which "operates" on the input vector function $\frac{\partial W_k(\tau, x_k(\tau, x_{ok}))}{\partial \tau}$. The kernel operator acts as a momentum averaging and diffusion effect. The vector function $W_k(\tau, x_k(\tau, x_{ok}))$,is given by (Eq. 7) a Newton potential of the external forces and pressure gradient divided by constant density, ρ_o, within the spherical control volume$V(\tau)$, centered at coordinate,$x_k(\tau, x_{ok})$.