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On Solutions to the Diophantine Equation $p^2 + q^2 = z^4$

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Abstract. In this paper, we investigate solutions to the title equation. It is established for all primes p, q that the equation has no solutions. The connection of the equation to Pythagorean triples $a^2 + b^2 = c^2$ is determined. In [7] all triples are presented where $5 \le c \le 2100$. All possible values c where $c \le 2100$ are examined, and the first 36 solutions of the equation when $z \le 45$ are established and also exhibited.

Keywords: Diophantine equations

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The field of Diophantine equations is ancient, vast and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving primes, composites and powers of all kinds. Among them, a minute fraction are [1, 2, 6].

In this paper, we consider the equation

$$p^2 + q^2 = z^4 \tag{1}$$

where p, q, z are positive integers. In Section 2, it is established for all primes p, q that equation (1) has no solutions. In Section 3, the connection between equation (1) and the Pythagorean triples is discussed. For all values $z \le 45$, it is established that equation (1) has exactly 36 solutions all of which are exhibited.

2. The equation $p^2 + q^2 = z^4$ is insolvable when p, q are primes

This result is shown in the following Theorem 2.1.

Theorem 2.1. If $p \geq 2$ and q are distinct primes, then $p^2 + q^2 = z^4$ has no solutions.

Proof: First, we consider the case $p = 2$, and then all odd primes p.

Suppose that $p = 2$ and q is prime. From (1) we have

$$4 = z^4 - q^2 = (z^2 - q)(z^2+q) \qquad z \text{ is odd.} \tag{2}$$

It follows from (2) that $z^2 – q = 2M$ and $z^2+q = 2N$ where $M \neq N$ are integers. Then, (2) yields

$$4 = (2M)(2N) \qquad \text{or} \qquad M \cdot N= 1 \quad \text{implying} \quad M = N = 1$$

which is impossible.

Hence, when $p = 2$, the equation $p^2 + q^2 = z^4$ has no solutions as asserted.

Suppose that p, q are odd primes. From (1) we obtain

$$p^2 = z^4 - q^2 = (z^2 - q)(z^2+q) \qquad z \text{ is even.} \tag{3}$$

Since p is prime, therefore $z^2 – q = 1,\ p,\ p^2, (z^2 + q = p^2,\ p, 1)$, where $z^2 – q = p,\ p^2$ are a priori eliminated. Thus, $z^2 – q = 1$ or $z^2 = q + 1$ and $2q + 1 = p^2$ implying that (3) yields

$$2q = p^2 – 1 = (p – 1)(p + 1). \tag{4}$$

In (4), $2 \mid (p – 1)$ and also $2 \mid (p + 1)$. It therefore follows that $(p – 1)(p + 1)$ is a multiple of at least 4, whereas $2q$ is a multiple of 2 only. Thus, (4) does not exist.

When p, q are odd primes, the equation $p^2 + q^2 = z^4$ has no solutions.

This concludes the proof of Theorem 2.1. □

3. The equation $p^2 + q^2 = z^4$ and Pythagorean triples

In this section we discuss the connection of the equation $p^2 + q^2 = z^4$ to the Pythagorean triples $a^2 + b^2 = c^2$.

A set of positive integers a, b, c is called a "Pythagorean triple" (abbreviated triple) denoted (a, b, c) if $a^2 + b^2 = c^2$.

The connection of Pythagorean triples to the equation $p^2 + q^2 = z^4$ is embedded as follows.

Set $p = a$, $q = b$ and $c = z^2$. Hence, whenever c equals a square, the equation $p^2 + q^2 = z^4$ has a solution which consists of a prime and a composite or of two composites.

In the following Table 1 we exhibit the first 36 solutions of the equation $p^2 + q^2 = z^4$. These are obtained from [7] "Pythagorean triples up to $c = 2100$" by considering all possible values $c = z^2$. The only two primes $p = 7$ and $p = 41$ respectively in **Solutions 1** and **19** are emphasized. All other integers p, q are composites.

Table 1: Solutions of $p^2 + q^2 = z^4$

Solution 1.	$7^2 + 24^2 = 5^4$
Solution 2.	$15^2 + 20^2 = 5^4$
Solution 3.	$28^2 + 96^2 = 10^4$
Solution 4.	$60^2 + 80^2 = 10^4$
Solution 5.	$65^2 + 156^2 = 13^4$
Solution 6.	$119^2 + 120^2 = 13^4$
Solution 7.	$63^2 + 216^2 = 15^4$
Solution 8.	$135^2 + 180^2 = 15^4$
Solution 9.	$136^2 + 255^2 = 17^4$
Solution 10.	$161^2 + 240^2 = 17^4$
Solution 11.	$112^2 + 384^2 = 20^4$
Solution 12.	$240^2 + 320^2 = 20^4$
Solution 13.	$175^2 + 600^2 = 25^4$
Solution 14.	$220^2 + 585^2 = 25^4$
Solution 15.	$336^2 + 527^2 = 25^4$
Solution 16.	$375^2 + 500^2 = 25^4$
Solution 17.	$260^2 + 624^2 = 26^4$
Solution 18.	$476^2 + 480^2 = 26^4$
Solution 19.	$41^2 + 840^2 = 29^4$
Solution 20.	$580^2 + 609^2 = 29^4$
Solution 21.	$252^2 + 864^2 = 30^4$
Solution 22.	$540^2 + 720^2 = 30^4$
Solution 23.	$544^2 + 1020^2 = 34^4$
Solution 24.	$644^2 + 960^2 = 34^4$
Solution 25.	$343^2 + 1176^2 = 35^4$
Solution 26.	$735^2 + 980^2 = 35^4$
Solution 27.	$444^2 + 1295^2 = 37^4$
Solution 28.	$840^2 + 1081^2 = 37^4$
Solution 29.	$585^2 + 1404^2 = 39^4$
Solution 30.	$1071^2 + 1080^2 = 39^4$
Solution 31.	$448^2 + 1536^2 = 40^4$
Solution 32.	$960^2 + 1280^2 = 40^4$
Solution 33.	$369^2 + 1640^2 = 41^4$
Solution 34.	$720^2 + 1519^2 = 41^4$
Solution 35.	$567^2 + 1944^2 = 45^4$
Solution 36.	$1215^2 + 1620^2 = 45^4$

Final remark. An interesting pattern is observed from the solutions contained in Table 1. When $z = 25$, exactly four solutions exist. For each and every other value z, exactly two solutions exist. Thus, in all 36 solutions, each value z occurs at least twice, and the solutions appear in pairs with respect to z. One may deduce, that for any value z which yields a solution of the equation, there exists at least another solution with the same value z. Hence, all solutions with the same value z occur in pairs.

We presume that the equation has infinitely many solutions in which p, q are composites.

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