Annals of Pure and Applied Mathematics
Vol. 18, No. 2, 2018, 207-212
ISSN: 2279-087X (P), 2279-0888(online)
Published on 28 December 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v18n2a11

Annals of **Pure and Applied Mathematics**

On Completion Problems for Various Subclasses of P_0^+ –Matrices

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Received 12 November 2018; accepted 23 December 2018

Abstract. In this paper, we study completions for weakly sign symmetric p_0^+-matrices, sign symmetric p_0^+-matrices and nonnegative p_0^+-matrices. We obtained that digraphs that include all loops and have weakly sign symmetric p_0^+-completion, sign symmetric p_0^+-completion and nonnegative p_0^+-completion are complete digraphs.

Keywords: Matrix completion, partial matrix, digraphs, weakly sign symmetric P_0^+-matrix, sign symmetric P_0^+-matrix, nonnegative P_0^+-matrix.

AMS Mathematics Subject Classification (2010): 15A48

1. Introduction

In this section we define terms and give a brief literature on related work.

Definition 1.1. A P**-matrix (**P_0**-matrix)** is a matrix in which every principal minor of the matrix A is positive (nonnegative) [1].

Definition 1.2. A $n \times n$ matrix is a P_0^+**-matrix** if for each $k \in \{1,...,n\}$, every $k \times k$ principal minor is nonnegative and at least one $k \times k$ principal minor is positive [2].

Clearly, P-matrix is both P_0-matrix and P_0^+-matrix. Also observe that P_0^+-matrix is a P_0-matrix.

Definitions 1.1 and 1.2 considers the values of the principal minors, the next definition gives restrictions on the type of entries of a matrix.

Definition 1.3. A $n \times n$ matrix $A = \lfloor a_{ij} \rfloor$ is

i. **Weakly sign symmetric(wss)** if $a_{ij}a_{ji} \geq 0$ for all i and j
ii. **Sign symmetric(ss)** if $a_{ij}a_{ji} > 0$ or $a_{ij} = a_{ji} = 0$ for all i and j

iii. **Nonnegative** if $a_{ij} \geq 0$ for all i and j

iv. **Positive** if $a_{ij} > 0$ for all i and j

Using Definition 1.3, we have four different subclasses of P_0^+-matrix (given in Definition 1.2).

Definition 1.4. A P_0^+-matrix A is called a **weakly sign symmetric P_0^+-matrix (resp. sign symmetric P_0^+-matrix)** if $a_{ij}a_{ji} \geq 0$ (resp. either $a_{ij}a_{ji} > 0$ or $a_{ij} = 0 = a_{ji}$) for all i and j. Similarly, A P_0^+-matrix A is called a **positive P_0^+-matrix (resp. nonnegative P_0^+-matrix)** if $a_{ij}a_{ji} > 0$ (resp. $a_{ij}a_{ji} \geq 0$) for all i and j.

Example 1.5. The matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -3 & 4 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ is a P_0^+-matrix since all principal minors are nonnegative and in every order there is at least one positive principal minor. Looking at the entries, it is clear that matrix A is a weakly sign symmetric P_0^+-matrix. It fails to be sign symmetric P_0^+-matrix because $a_{13} = 2 \neq a_{31} = 0$, again it is not a nonnegative P_0^+-matrix since both $a_{12} = -2$ and $a_{21} = -3$ are negatives and by the same fact it is not a positive P_0^+-matrix.

Definition 1.6. A $P_{0,1}^+$ **-matrix** is a P_0^+-matrix whose diagonal entries positive and a **positive $P_{0,1}^+$ -matrix** is a $P_{0,1}^+$ -matrix in which all entries are positive.

Proposition 1.7. A matrix is a positive $P_{0,1}^+$ -matrix if and only if it is positive P_0^+-matrix

Proof: Positive $P_{0,1}^+$ -matrix is a $P_{0,1}^+$ -matrix in which all entries are positive (from Definition 1.6), it means the condition that all diagonal entries are positive and hence it is a positive P_0^+-matrix.

Conversely, a positive P_0^+-matrix is a P_0^+-matrix in which all entries are positive hence all diagonal entries are positive, therefore it correct to say it is a positive $P_{0,1}^+$ -matrix (although diagonal entries have been repeatedly been mentioned to be positive).

Definition 1.8. A **partial matrix** is a matrix in which some entries are specified while others are free to be chosen. Let $\prod$ be a class of matrices (e.g. weakly sign symmetric P_0^+ -, sign symmetric P_0^+ -, nonnegative P_0^+ - and positive P_0^+ -matrices) then a partial $\prod$ -matrix is one whose specified entries satisfy the required properties of a $\prod$ -matrix.

Graph theoretic approach will be used in completing these partial matrices, and some definitions are given as follows.

Definition 1.9. A **digraph** $D=(V_D, E_D)$ is a graph G with ordered pairs (u,v) of vertices and arc where u the initial vertex is and v is the terminal vertex. The order of a digraph D denoted n is the number of vertices of D. A digraph is **complete digraph** if it includes all possible arcs between its vertices (also called **clique**) [3].

A $n \times n$ partial matrix A is said to **specify** a digraph D on vertices $\{v_1,...,v_n\}$ if (v_i, v_j) is an arc in D if and only if the entry a_{ij} of A is specified.

Definition 1.10. A **completion** of a partial matrix is a specific choice of values for the unspecified entries. If we consider classes given in Definition 2.8, a digraph D has $\prod$ -completion if any partial $\prod$ -matrix specifying D can be completed to a $\prod$ -matrix.

On the related work, we just give a brief history of matrix completions close to our research class. Research on P -matrix completion was first studied by Johnson and Kroschel in [4] and later extended by DeAlba and Hogben in [5]. In 2003, a subclass of P -matrices: weakly sign symmetric P -matrices was studied in [6] and then in two subclasses: positive and nonnegative P -matrices were considered in [7]. Another class on P_0 -matrices was investigated first by Choi and others in [1], and its subclasses, weakly sign symmetric P_0 -matrices, nonnegative symmetric P_0 -matrices and sign symmetric P_0 -matrices were consider in the following papers [6], [8] and [9] respectively. In 2015, a new class of P_0^+ -matrices was first introduced and classification of digraphs of up to order 4 having P_0^+ -completion was done. It is in this class that we are interested in, and the subclasses to be discussed are weakly sign symmetric P_0^+ -matrices, sign symmetric P_0^+ -matrices and nonnegative P_0^+ -matrices.

2. Preliminaries

In this section, we will present some basic results that will be useful in the next section.

If a partial wss P_0^+ -matrices, ss P_0^+ -matrices and nonnegative P_0^+ -matrices omits all diagonal entries then it can be completed to wss P_0^+ -matrices, ss P_0^+ -matrices and nonnegative P_0^+ -matrices by assigning sufficiently large values to unspecified diagonal entries. In this research we are interested in the situations where all diagonal entries are specified. Zeros along diagonal entries tend to make completion for the three subclasses difficult.

Consider $A = \begin{bmatrix} x & 1 \\ 2 & 0 \end{bmatrix}$ which is a partial wss P_0^+ -matrix, partial ss P_0^+ -matrix and a partial nonnegative P_0^+ -matrix specifying digraph in Fig. 2.1 and cannot be completed to a wss P_0^+ -matrix, a ss P_0^+ -matrix and a nonnegative P_0^+ -matrix respectively since $\det A = -2 < 0$ for any value of x . Thus the digraph in Figure 2.1 does not have wss P_0^+ -completion, ss P_0^+ - completion and nonnegative P_0^+ - completion

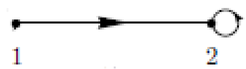

Figure 2.1:

Now, in the next section we assume that all digraphs have diagonal entries specified.

3. Main results

Our main results on completions of various subclasses of P_0^+ - matrices namely weakly sign symmetric P_0^+ - matrices, sign symmetric P_0^+ -matrices and nonnegative P_0^+ -completion are presented in Theorem 3.1, 3.2 and 3.3 respectively.

Theorem 3.1. The digraphs having all loops and weakly sign symmetric P_0^+ -completion are complete digraph.

Proof: Let wss $n \times n$ P_0^+ -matrix A_c be a completion of partial wss $n \times n$ P_0^+ -matrix A having all diagonal entries specified. Assume that the partial wss $n \times n$ P_0^+ -matrix A has the first $n-1$ diagonal entries as 0 and the last is 1 with specified entries a_{ij}'s and unspecified entries x_{ij}'s . Consider the 2×2 principal minors $\det A(i, j)$ for some $i, j \in \{1,...,n\}$. Note that $d_i d_j = 0$ always. Now split into three cases:

Case 1: Position ij and ji are specified. In this case we have

$$\det A(i, j) = d_i d_j - a_{ij} a_{ji} = -a_{ij} a_{ji} \geq 0$$

Thus $a_{ij} a_{ji} \leq 0$ and by wss P_0^+ -completion ($a_{ij} a_{ji} \geq 0$) we have $a_{ij} a_{ji} = 0$

Case 2: Position ij is specified and ji is unspecified. In this case we have

$$\det A(i, j) = d_i d_j - a_{ij} x_{ji} = -a_{ij} x_{ji} \geq 0$$

Thus $a_{ij}x_{ji} \le 0$ and by wss P_0^+-completion we have $a_{ij}x_{ji} = 0$

Case 3: Position ij and ji are unspecified. In this case we have

$$\det A(i, j) = d_i d_j - x_{ij}x_{ji} = -x_{ij}x_{ji} \ge 0$$

Thus $x_{ij}x_{ji} \le 0$ and by wss P_0^+-completion we have $x_{ij}x_{ji} = 0$

Observe that in all cases the product of twin entries is zero. However wss P_0^+-completion requires that at least one of 2×2 principal minors is positive. This is a contradiction.

Theorem 3.2. The digraphs having all loops and sign symmetric P_0^+-completion are complete digraphs.

Proof: Using same hypothesis as in Theorem 3.1, again consider the 2×2 principal minor $\det A(i, j)$ for some $i, j \in \{1,...,n\}$ and that $d_i d_j = 0$ always. This means if a non-diagonal entry is specified then it must be zero (0) that is $a_{ij} = 0$ since $\det A(i, j) = d_i d_j - a_{ij}x_{ji} = -x_{ij}a_{ji} < 0$ if $a_{ij} \ne 0$. Therefore all unspecified non-diagonal twin entries x_{ji} are also assigned zero (0) that is $c_{ji} = 0$. As a result all non-diagonal entries have zeros hence $\det A(\alpha) = 0 \ \ \forall \alpha \in \{1,...,n\}$ this shows that partial ss P_0^+-matrices with unspecified entries lack sign symmetric P_0^+-completion and so, the only digraphs having all loops and sign symmetric P_0^+-completion are complete digraphs.

Theorem 3.3. The digraphs having all loops and nonnegative P_0^+-completion are complete digraph.

Proof: The proof for this theorem follows from the proof of Theorem 3.1, also having three cases with all specified entries a_{ij} s being nonnegative i.e. $a_{ij} \ge 0$ and values assigned to unspecified entries x_{ij} s being nonnegative that is $c_{ij} \ge 0$, which also shows that all the three cases have the product of the twin entries being zero and similar to Theorem 3.1 does not have 2×2 principal sub-matrix with positive determinant hence digraphs having all loops and nonnegative P_0^+-completion are complete digraph.

4. Conclusion and recommendations

Based on the main results we have concluded that digraphs that include all loops and have weakly sign symmetric P_0^+-completion, sign symmetric P_0^+-completion and nonnegative P_0^+-completion are complete digraphs. According to sections on related work and main results, we observe that similar research should be done for positive P_0^+-matrices.

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