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# On the Diophantine Equation $\{(q^2)^n\}^x + p^y = z^2$ where $q$ is any Prime Number and $p$ is an Odd Prime Number

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***Abstract.*** In this paper, we have solved the Diophantine equation $\left\{(3^2)^n\right\}^x + p^y = z^2$ and $\left\{(5^2)^n\right\}^x + p^y = z^2$ where $n \in Z^+$ and p is an odd prime. Also, we have discussed the generalization of $(4^n)^x + p^y = z^2$ to $\left\{(q^2)^n\right\}^x + p^y = z^2$, where $n \in Z^+$, q is any prime number and p is an odd prime number. Some solutions of these Diophantine equations have been obtained.

***Keywords:*** Diophantine equations, Exponential Diophantine equations and Catalan's conjecture

***AMS Mathematics Subject Classification (2010):*** 11D61

## 1. Introduction

Diophantine equations are central objects of number theory in mathematics with a vital importance in the field of Cryptography, Computer Science, Chemistry, Geometry and many more. According to Cao [4], the Diophantine equation $a^x + b^y = c^z$ has at most one solution for $z$>1. Suvarnamani et al. [9] proved that the Diophantine equation $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have not any non-negative integer solution. Chatchaisthit [5] have presented that the Diophantine equation$4^x + p^y = z^2$ have the solutions of the form $(x,p,y,z) \in \{(2,3,2,5)\} \cup \left\{(r, 2^{r+1} + 1, 1, 2^r + 1): r \in N \cup \{0\}\right\} \cup \left\{(r, 2, 2r + 3, 3.2^r): r \in N \cup \{0\}\right\}$ where $p$ is a prime number. Peker and Cenberci [8] worked on the Diophantine equation $(4^n)^x + p^y = z^2$ and the obtained solutions are $(x,y,z,p) = (1,2,5,3), (2,2,5,3)$and $(k,1,2^{nx}+1, 2^{nx+1})$, where $k$ is a non-negative integer and $p$ is an odd prime number , $n \in Z^+$. Burshtein [1] discussed the conditions for the solution of Diophantine equation $p^x + q^y = z^2$ based on the various values of p and q where p, q both are prime such that $p < q$ and differ by an even value $k$. Burshtein [2] discussed and found that the Diophantine equation $p^x + q^y = z^2$ has infinitely many solutions when $p$ = 2, 3 and also demonstrated that if prime $p$> 3 than the equation has a solution for each and every integer $x$≥1. Burshtein [3] discussed all the solutions to an open problem of Chotchaisthit on the Diophantine equation $2^x + p^y = z^2$ when $y$ = 1 and $p$ = 7, 13, 29, 37,

257. Kumar, Gupta and Kishan [6] solved the Diophantine equation $61^x+67^y=z^2$ and $67^x+73^y=z^2$ and proved that the equations have not any non-negative integer solution.
In this study, we discuss the Diophantine equation $\{(q^2)^n\}^x + p^y = z^2$ where $q$ is any prime number and $p$ is an odd prime number. We will use the Catalan's conjecture [7] and factor method to solve this Diophantine equation.

## 2. Preliminary

**Lemma 2.1.** Catalan's conjecture state that the only solution of the Diophantine equation $a^x - b^y = 1$ is $(a,b,x,y) = (3,2,2,3)$ with $a>1, b>1, x>1$, and $y>1$.

**Lemma 2.2.** If $p$ is an odd prime &$n \geq 2$ is integer, than $x^2 - 1 = p^n$has no solutions [7].

**Lemma 2.3.** The Diophantine equation $\{(2^2)^n\}^x + p^y = z^2$ has the solutions $(x,y,z,p) = (1,2,5,3), (2,2,5,3)$ and $(k,1,2^{nx}+1, 2^{nx+1})$ where $k$ is a non-negative integer [5].

## 3. Main theorems

**Theorem 3.1.** The Diophantine equation $\{(3^2)^n\}^x + p^y = z^2$, has the solution $(x, y, z, p) = (k, 1, 3^{nk}+1, 2.3^{nk}+1)$, where $p$ is an odd prime, $k \geq 0, n \in Z^+$ and $x, y, z \in Z^+ \cup \{0\}$.

**Proof.** We consider the Diophantine equation

$$\{(3^2)^n\}^x + p^y = z^2, \tag{1}$$

where $n \in Z^+$ and $x$,$y$ and $z$ are non-negative integers.

Now we discuss this problem in three cases.

**Case 1.** For $n$=1, equation (1) is given as

$$\{(3^2)^1\}^x + p^y = z^2$$

Or
$$(3^2)^x + p^y = z^2. \tag{2}$$

Now for $y > 0$,
$$(3^2)^x + p^y = z^2$$

Or
$$z^2 - (3^2)^x = \mathrm{p}^y$$

Or
$$(z - 3^x)(z + 3^x) = p^y.$$

This implies that, $z - 3^x = p^v, \quad z + 3^x = p^{y-v}$ , where $y > 2v$.

Now, we have
$$p^{y-v} - p^v = 2.3^x$$

Or
$$p^v(p^{y-2v} - 1) = 2.3^x.$$

For *v*=0, $p^y - 1 = 2.3^x$

Or $p^y = 2.3^x + 1.$

For *y*=1, $p = 2.3^x + 1$

and $z = 3^x + 1.$

Therefore we get (*x*, *y*, *z*, *p*) =(*k*, 1,$3^k + 1$, $2.3^k + 1$), where *k* is non-negative integer.

For *x*=0, the equation (1) can be written as

$$(3^2)^0 + p^y = z^2$$

Or $z^2 - p^y = 1.$

By Lemma 2.1, it has no solution for *p* an odd prime.

Now, for *y* = 0, then the equation (2) can be written as

$$z^2 - 3^{2x} = 1$$

Therefore, it has no solution for p is an odd prime$z^2 - 1 = 3^{2x}$has no solution by Lemma 2.2.

**Case 2.** for n=2, equation (1) is given as

$$\{(3^2)^2\}^x + p^y = z^2$$

Or $$(3^4)^x + p^y = z^2. \quad (3)$$

Now for *y* > *0*, $(3^4)^x + p^y = z^2$

Or $z^2 - (3^4)^x = p^y$

Or $(z - 3^{2x})(z + 3^{2x}) = p^y.$

This implies that, $z - 3^{2x} = p^v$, $z + 3^{2x} = p^{y-v}$ , where $y > 2v$

Now we have $p^{y-v} - p^v = 2.3^{2x}$

Or $p^v(p^{y-2v} - 1) = 2.3^{2x}.$

For *v*=0, $p^y - 1 = 2.3^{2x}$

Or $p^y = 2.3^{2x} + 1.$

For *y*=1, $p = 2.3^{2x} + 1$ and $z = 3^{2x} + 1$

Therefore we get (*x*, *y*, *z*, *p*) = (*k*, 1, $3^{2k} + 1$, $2.3^{2k} + 1$), where *k* is non-negative integer.

For *y*=0 same as case 1, and for *x*=0 same as case 1.

**Case 3.** for all *n*$\in Z^+$ equation (1) is given as

$$\{(3^2)^n\}^x + p^y = z^2. \quad (4)$$

Now for $y > 0$, $(3^{2n})^x + p^y = z^2$

Or $z^2 - (3^{2n})^x = p^y$

Or $(z - 3^{nx})(z + 3^{nx}) = p^y$.

This implies that $z - 3^{nx} = p^v, \quad z + 3^{nx} = p^{y-v}$, where $y > 2v$

Now we have $p^{y-v} - p^v = 2.3^{nx}$

Or $p^v(p^{y-2v} - 1) = 2.3^{nx}$.

For $v$=0, $p^y - 1 = 2.3^{nx}$

Or $p^y = 2.3^{nx} + 1$.

For $y$=1, $p = 2.3^{nx} + 1$ and $z = 3^{nx} + 1$

Therefore we get ($x$, $y$, $z$, $p$) =($k$, 1, $3^{nk} + 1$, $2.3^{nk} + 1$),where $k$ is non- negative integer.

For $y$=0 same as case 1, and for $x$=0 same as case 1.

Hence The Diophantine equation$\{(3^2)^n\}^x + p^y = z^2$, has the solution ($x$, $y$, $z$, $p$) = ($k$, 1, $3^{nk}$+1, $2.3^{nk}$+1), where $p$ is an odd prime, $k \geq 0$,$n \in Z^+$ and $x, y, z \in Z^+ \cup \{0\}$.

**Theorem 3.2.** The Diophantine equation $\{(5^2)^n\}^x + p^y = z^2$ has the solution($x$, $y$, $z$, $p$) = ($k$, 1, $5^{nk}$+1, $2.5^{nk}$+1), where $p$ is an odd prime, $k \geq 0$, $n \in Z^+$and $x, y, z \in Z^+ \cup \{0\}$.

**Proof:** The Diophantine equation

$$\{(5^2)^n\}^x + p^y = z^2, \quad (5)$$

where $n \in Z^+$and $x, y, z \in Z^+ \cup \{0\}$

Now for $y > 0$, $(5^{2n})^x + p^y = z^2$

Or $z^2 - (5^{2n})^x = p^y$

Or $(z - 5^{nx})(z + 5^{nx}) = p^y$.

This implies that, $z - 5^{nx} = p^v, \quad z + 5^{nx} = p^{y-v}$, where $y > 2v$

We have $p^{y-v} - p^v = 2.5^{nx}$

Or $p^v(p^{y-2v} - 1) = 2.5^{nx}$.

For $v$=0, $p^y - 1 = 2.5^{nx}$

Or $p^y = 2.5^{nx} + 1$.

For $y$=1, $p = 2.5^{nx} + 1$ and $z = 5^{nx} + 1$

Thus we get ($x$, $y$, $z$, $p$) = ($k$, 1,$5^{nk} + 1$, $2.5^{nk} + 1$) , where $k$ is non- negative integer.

For $x$=0, $(5^{2n})^0 + p^y = z^2$

Or $$z^2 - p^y = 1.$$

By Lemma 2.1, it has no solution for $p$ is an odd prime.
Now, $y$=0, we have $$z^2 - 5^{2nx} = 1.$$

Therefore, $z^2 - 1 = 5^{2nx}$ has no solution by Lemma 2.2.

Hence the Diophantine equation $\{(5^2)^n\}^x + p^y = z^2$ has the solution($x$, $y$, $z$, $p$) = ($k$, 1, $5^{nk}$+1, $2.5^{nk}$+1), where $p$ is an odd prime, $k \geq 0$, $n \in Z^+$ and $x, y, z \in Z^+ \cup \{0\}$.

**Theorem 3.3.** The Diophantine equation $\{(q^2)^n\}^x + p^y = z^2$ has the solution ($x$, $y$, $z$, $p$) = ($k$, 1, $q^{nk}$+1, $2.q^{nk}$+1), where $q$ is any prime number, $p$ is an odd prime, $k \geq 0, n \in Z^+$ and $x, y, z \in Z^+ \cup \{0\}$.

**Proof:** The Diophantine equation

$$\{(q^2)^n\}^x + p^y = z^2, \tag{6}$$

where $n \in Z^+$ and $x, y, z \in Z^+ \cup \{0\}$

Now for $y > 0$, $$(q^{2n})^x + p^y = z^2$$

Or $$z^2 - (q^{2n})^x = p^y$$

Or $$(z - q^{nx})(z + q^{nx}) = p^y.$$

This implies that, $z - q^{nx} = p^v$, & $z + q^{nx} = p^{y-v}$, where $y > 2v$

We have $$p^{y-v} - p^v = 2.q^{nx}$$

Or $$p^v(p^{y-2v} - 1) = 2.q^{nx}.$$

For $v$=0, $$p^y - 1 = 2.q^{nx}$$
Or $$p^y = 2.q^{nx} + 1.$$
For $y$=1, $p = 2.q^{nx} + 1$ and $z = q^{nx} + 1$.
Therefore we get ($x$, $y$, $z$, $p$) = ($k$, $1, q^{nk} + 1, 2.q^{nk} + 1$), where $q$ is any prime number, $p$ is an odd prime and $k$ is non-negative integer.
For $x$= 0, $$(q^{2n})^0 + p^y = z^2$$

Or $$z^2 - p^y = 1.$$

By Lemma 2.1, it has no solution for $p$ is an odd prime.
Now, for $y$=0, we have $$z^2 - q^{2nx} = 1.$$

Therefore, $z^2 - 1 = q^{2nx}$ has no solution by Lemma 2.2.

Hence The Diophantine equation $\{(q^2)^n\}^x + p^y = z^2$ has the solution ($x, y, z, p$) = ($k$, 1, $q^{nk}+1$, $2.q^{nk}+1$), where $q$ is any prime number, $p$ is an odd prime, $k \geq 0$, $n \in Z^+$ and $x, y, z \in Z^+ \cup \{0\}$.

## 4. Conclusion

In this paper, we find out the solution ($x$, $y$, $z$, $p$) = ($k$, 1, $q^{nk}+1$, $2.q^{nk}+1$) of the Diophantine equation $\{(q^2)^n\}^x + p^y = z^2$, where $p$ is an odd prime and $k \geq 0$.

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