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Heterogeneous Server Markovian Queue with Partial Breakdown

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Abstract. In this paper we consider a two heterogeneous server Markovian queue with partial breakdown. If both the servers are ideal, the arriving customer select the fastest server for service. During busy period the system may breakdown, immediately repair has been carried out. But, during the breakdown period the server work in a slower rate. This model has been solved using Matrix geometric method. Some performance measures and numerical results are obtained.

Keywords: Markovian queue, Heterogeneous server, Steady state solution, Matrix-Geometric method

AMS Mathematics Subject Classification (2010): 90B22, 60K25, 60K30

1. Introduction

Queueing system with breakdown / partial breakdown has applications in manufacturing system, production system, telecommunication network and computer system. Single sever queueing systems with server breakdown have been studied by many researchers including Federgruen and Green [4], Li et al. [12], Tang [24], Nakdimon and Yechiali [17], Wang et al. [28], Wang et al. [27], Choudhury and Tadj [2], to mention a few. Multi-server queueing systems with server breakdowns are more flexible and applicable in practice than the single server counterpart. However, due to their analytical complexity, there have been only a few studies carried out on multi-server queueing systems with server breakdown. The equilibrium analysis for the general input and exponential service time and with n servers was given in Kendall [7]. A non-constructive existence theorem for the stationary distribution of a general input and general service time was presented in Kiefer and Wolfowitz [8]. Karlin and Mc Gregor [6] obtained the busy period distribution for the $M/M/S$queue. Krishnamoorthi [9] considered a Poisson queue with two heterogeneous severs and with violation of the First-in-First-out principle. Mitrany and Avi-Itzhak [16], studied an $M/M/N$ queue with server breakdown and ample repair capacity. In their study, the moment generating function of the queue size has been obtained by using the transformation method.

Heffer [5] has analyzed waiting time distribution of $M/E_k/S$queue. Sing [22] studied an $M/M/2$ queueing system with balking and heterogeneous servers. In 1970, the author obtained the stationary queue length distribution and the mean queue length and also compared the model with heterogeneous servers and the model with homogeneous servers. Singh [23] discussed a Markovian queue with the number of servers depending upon the queue length. Desmit [3] presented an approach to identify the distribution of waiting times and queue lengths for the queue$GI/H_2/S$. He reduced the problem to the solution of the Wiener-Hopf-type equations and then used a factorization method to solve the system. Lin and Kumar [13] has analyzed the optimal control of a queuing system with two heterogeneous servers. Rubinovitch [21] studied the problem of a heterogeneous two channels queuing systems. In his first paper he discussed three simple models and gave the condition when to discard the slower server depending on the expected number of customers in the system. In the second paper he studied a queuing model with a stalling concept.

Vinod [25], considered the model of Mitrany and Avi-Itzhak [16] using the matrix-geometric solution method. For $N = 1$, the author imposed some restrictions on the server down-periods (either independent of the queue length or only occurring when the sever is active). Neuts and Lucantoni [20] and Wartenhosrt [30] studied the repair model by considering limited repair capacity. Neuts and Lucantoni [20], considered a single queue of customers, each served by one of N parallel servers. Wang and Chang [27] studied an $M/M/R/N$ queue with balking, reneging and server breakdowns from the viewpoint of queueing. They solved the steady-state probability equations iteratively and derived the steady-state probabilities in matrix form.

In 1999, Abou-El-ata and Shawky [1] introduced a simpler approach to find the condition when to discard the slower server in a heterogeneous two channels queue. Kumar and Madheswari [10], studied an $M/M/2$ queueing system with heterogeneous servers and multiple vacations by using the matrix-geometric solution method. They studied the stationary queue length distribution and waiting time distribution along with their means via the rate matrix. Yue et al. [31], further considered the model in 2005. They obtained the explicit expression of the rate matrix and proved the conditional stochastic decomposition results for the stationary queue length and waiting time. Madan et al. [15], studied a two-server queue with Bernoulli schedules and a single vacation policy where the two servers provide heterogeneous exponential services to the customers. They obtained steady-state probability generating functions of the system size for various states of the servers. Rani and Shanthi [21] studied M/M/2/K with controllable arrival rate.

In this paper we consider a two heterogeneous server Markovian queue with partial breakdown. If both the servers are ideal, the arriving customer selects the fastest server. During busy period the system may breakdown, immediately repair has been carried out. But, during the breakdown period the servers work in a slower rate. The model has been defined in section 2 and the model has been analyzed in section 3. Some performance measures are given in section 4, a numerical study has been carried out in section 5. A conclusion has been given in the last section.

2. The model

We consider an $M/M/2$ queueing model with heterogeneous servers, server 1 and server 2. The inter arrival time of customers follows negative exponential distribution with mean $\frac{1}{\lambda}$. Server 1 and 2, served the customers based on exponential distribution with rates $\mu_1$and μ_2 respectively $(\mu_2 < \mu_1)$ and let $\mu = \mu_1 + \mu_2$. Each customer is served only by one server and the queue discipline is first come first served. If the system is empty arriving customer always joins the first server. During service, the system may breakdown, the breakdown period follows negative exponential distribution with rate α. Immediately the repairing process starts, the repair period follows negative exponential distribution with rate β. During the breakdown period, the servers serves the customers with lower rates μ_3, μ_4 respectively and let $\mu' = \mu_3 + \mu_4$ $(\mu_1 > \mu_2 > \mu_3 > \mu_4)$. This behavior of the system is called as system with partial breakdown, if an arriving customer finds both the servers busy the customer waits in a waiting line of a infinite capacity for the first free server.

3. The analysis

At time t, let L(t) be the number of customers in the system and J(t) the server state.

$$J(t) = \begin{cases} 0, if\ the\ server\ is\ in\ partial\ breakdown \\ 1, \qquad\qquad if\ the\ server\ is\ in\ busy\ state \end{cases}$$

Let $X(t) = (L(t), J(t))$, then $\{(X(t)): t \geq 0\}$ is a Continuous time Markov chain (CTMC) with state space$S = \{(i,j): i = 0,1; j \geq 0\}$, where j denotes the number of customer in the system and i denotes the server state.

$$Q = \begin{bmatrix} C_0 & C_1 & & & & & \\ B_0 & B_1 & A_0 & & & & \\ & A_2 & A_1 & A_0 & & & \\ & & A_2 & A_1 & A_0 & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & \ddots \end{bmatrix}$$

where the sub-matrices A_0, A_1, and A_2 are of order 2 x 2 and are appearing as

$$A_0 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -(\lambda + \mu' + \beta) & 0 \\ 0 & -(\lambda + \mu) \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \mu' & 0 \\ 0 & \mu \end{bmatrix}$$

and the boundary matrix is defined by

$$C_0 = -(\lambda)$$

$$C_1 = (0 \quad \lambda)$$

$$B_0 = \begin{bmatrix} 0 \\ \mu_1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -(\lambda + \beta) & \beta \\ \alpha & -(\lambda + \alpha + \mu_1) \end{bmatrix}$$

Let $P = (p_0, p_1, p_2, \ldots)$ be the stationary probability vector associated with Q, such that PQ=0 and Pe=1, where e is a column vector of 1's of appropriate dimension, where $p_0 = (p_0)$, $p_i = (p_{i0}, p_{i1})$ for $i \geq 1$.

If the steady state condition is satisfied, the sub vectors p_i satisfies following equations:

$p_0 C_0 + p_1 B_0 = 0$ (1)

$p_0 C_1 + p_1 B_1 + p_2 A_2 = 0,$ (2)

$p_i A_0 + p_{i+1} A_1 + p_{i+2} A_2 = 0, i \geq 1$ (3)

$p_i = p_1 R^{i-1}; i \geq 2$ (4)

where R is the rate matrix, is the minimal non-negative solution of the matrix quadratic equation (see Neuts [19]).

$R^2 A_2 + R A_1 + A_0 = 0$ (5)

Substituting the equation (4) in (2), we have

$p_0 C_1 + p_1 (B_1 + R A_2) = 0$ (6)

and the normalizing condition is

$p_0 + p_1 (I - R)^{-1} e = 1$ (7)

Theorem 1. The queueing system described in this article is stable if and only if $\boldsymbol{\rho}<1$, where $\rho = \frac{\lambda(\alpha+\beta)}{\alpha\mu'+\beta\mu}$

Proof: Consider the infinitesimal generator $A = \begin{bmatrix} -\beta & \beta \\ \alpha & -\alpha \end{bmatrix}$, which is a square matrix of order 2, the row vector $\pi = (\pi_0, \pi_1)$ satisfying the condition $\boldsymbol{\pi}$A=0 and $\boldsymbol{\pi}$e=1.

That is,

The system is stable if and only if $\frac{\lambda(\alpha+\beta)}{\alpha\mu'+\beta\mu} < 1$

Theorem 2. If $\boldsymbol{\rho}$<1, the matrix equation (5) has the minimal non-negative solution $R = -A_0 A_1^{-1} - R^2 A_2 A_1^{-1}$

Proof: We define the matrix $A = A_0 + A_1 + A_2$. This matrix A is a 2 x 2 matrix and it can be written as A= $\begin{bmatrix} -\beta & \beta \\ \alpha & -\alpha \end{bmatrix}$

A is reducible. The analysis present in Neuts [18] is not applicable. In Lucantoni [14], similar reducible matrix is treated for the case when the elements are probabilities.

Equation (5), can be written as,

$A_0 A_1^{-1} + R A_1 A_1^{-1} + R^2 A_2 A_1^{-1} = 0 A_1^{-1}$

Since A_1 is non-singular, A_1^{-1} exists and

$R = -A_0 A_1^{-1} - R^2 A_2 A_1^{-1}$ (8)

where

$$A_1^{-1} = \frac{1}{[(\lambda+\beta+\mu)(\lambda+\beta+\mu') - \alpha\beta]} \begin{bmatrix} -(\lambda+\beta+\mu) & -\beta \\ -\alpha & -(\lambda+\beta+\mu') \end{bmatrix}$$

Using Neuts and Lucantoni [20], the matrix R is numerically computed by using the recurrence relation with R(0)=0 in equation (8).

Theorem 3. If $\boldsymbol{\rho}$<1, the stationary probability vectors p_0 and $p_i = (p_{i0}, p_{i1})$ are

$$p_0 = \frac{1}{\left[1 + \left(\frac{\lambda}{(1-r_0)(1-r_1) - r_{10} r_{01}}\right)\left(\frac{(1 - r_1 + r_{01})(\mu' r_{10} + \alpha)}{\mu_1(\lambda+\beta+\mu' r_0)} + \frac{1 - r_0 + r_{10}}{\mu_1}\right)\right]}$$

$$p_{10} = \left[\frac{\lambda(\mu' r_{10} + \alpha)}{\mu_1(\lambda+\beta-\mu' r_0)}\right] p_0$$

$p_{11} = \left(\frac{\lambda}{\mu_1}\right) p_0$
and $p_i = p_1 R^{i-1}; i \geq 2$
Proof: p_0, p_{10} and p_{11} follows from the equations (1), (2) and (7).

Remark 1. Even though R in Theorem 2 has a nice structure which enables us to make use of the properties like$R^n = \begin{bmatrix} r_0^n & r_{01} \sum_{j=0}^{n-1} r_0^j r_1^{n-j-1} \\ 0 & r_1^n \end{bmatrix}$, for $n \geq 1$due to the form of r_0 & r_{01}, it may not be easy to cary out the computation required to calculate the p_i and the performance measures. Hence, we explore the possibility of algorithmic computation of R. The computation of R can be carried out using a number of well-known methods. We use Theorem 1 of Latouche and Neuts [11]. The matrix R is computed by successive substitutions in the recurrence relation:

$R(0) = 0$ (9)

$R(n+1) = -A_0 A_1^{-1} - [R(n)]^2 A_2 A_1^{-1}$ for $n \geq 0$ (10)

and is the limit of the monotonically increasing sequence of matrices $\{R(n), n \geq 0\}$.

4. Performance measures

Using straightforward calculations the following performance measures have been obtained:

(i) probability of down time = p_0
(ii) probability that both the servers be busy = $1 - p_0 - (p_{10} + p_{11})$
(iii) Expected number of customers in the system $E(N) = \sum_{n=0}^{\infty} n p_n e$
(iv) Expected number of customer served $E(M) = \mu_1 p_1 e + (\mu_1 + \mu_2) \sum_{n=2}^{\infty} p_n e$
(v) Expected waiting time of a customer in the system, according to Little's law, is $E(W) = \frac{E(N)}{\lambda}$

5. Numerical study

In this section, some examples are given to show the effect of the parameters λ, μ_1, μ_2, μ_3, μ_4, α and β on the probability of down time, probability that both the servers be busy, Expected number of customers in the system, Expected number of customer served, Expected waiting time of a customer in the system for the model analyzed in this paper. The corresponding results are presented as case (1), case (2).

Case (1): If λ=0.3, $\mu_1 = 4.5$, $\mu_2 = 3.2$, $\mu_3 = 2.5$, $\mu_4 = 1.2$, α=0.2 and β=0.5, the matrix R is obtained using the equations (9) & (10)

$R = \begin{bmatrix} 0.071038 & 0.004826 \\ 0.001859 & 0.038068 \end{bmatrix}$

and the invariant probability vector is
$P = (p_0, p_1, p_2, \ldots)$ where
$p_0 = (0.911280)$
$p_1 = (0.023421, 0.060812)$
and the remaining vectors $p_i's$ are evaluated using the relation $p_i = p_1 R^{i-1}; i \geq 2$
$p_2 = (0.001776830526, 0.002428021049)$
$p_3 = (0.000130736167, 0.000101004887)$

$p_4 == (0.000009475004, 0.000004475987)$
$p_5 = (0.000000681406, 0.000000216118)$
$p_6 = (0.000000048807, 0.000000011516)$
$p_7 = (0.000000003489, 0.000000000674)$
$p_8 = (0.000000000249, 0.000000000042)$
$p_9 = (0.000000000018, 0.000000000003)$

For the chosen parameters $p_9 \to 0$, and the sum of the steady state probabilities is found to be 0.999964.

The performance measures are

(i) probability of down time $p_0 = 0.911280$
(ii) probability that both the servers be busy =0.004451
(iii) Expected number of customers in the system E(N)=0.0933985
(iv) Expected number of customer served E(M)=0.4133212
(v) Expected waiting time of a customer in the system, according to Little's law, is E(W)=0.311328

Case (2): If λ=0.5, $\mu_1 = 5.2$, $\mu_2 = 4.6$, $\mu_3 = 3.3$, $\mu_4 = 2.2$, α=0.3 and β=0.6, the matrix R is obtained using the equations (9) & (10)

$$R = \begin{bmatrix} 0.081549 & 0.005253 \\ 0.002531 & 0.049600 \end{bmatrix}$$

and the invariant probability vector is
$P = (p_0, p_1, p_2, ...)$ where
$p_0 = (0.867201)$
$p_1 = (0.040272, 0.083577)$
and the remaining vectors $p_i's$ are evaluated using the relation $p_i = p_1 R^{i-1}; i \geq 2$
$p_2 = (0.003495675046, 0.004356968217)$
$p_3 = (0.000296096317, 0.000234468418)$
$p_4 = (0.000024739800, 0.000013185027)$
$p_5 = (0.000002050877, 0.000000783936)$
$p_6 = (0.000000169231, 0.000000049656)$
$p_7 = (0.000000013926, 0.000000003352)$
$p_8 = (0.000000001144, 0.000000000239)$
$p_9 = (0.000000000094, 0.000000000018)$
$p_{10} = (0.000000000008, 0.000000000001)$
For the chosen parameters $p_{10} \to 0$, and the sum of the steady state probabilities is found to be 0.999474.

The performance measures are

(i) probability that down time p_0= 0.867201
(ii) probability that both the servers be busy =0.008424
(iii) Expected number of customers in the system E(N)=0.141312
(iv) Expected number of customer served E(M)=0.72657
(v) Expected waiting time of a customer in the system, according to Little's law, is E(W)=0.282624

6. Conclusion

In this article, a heterogeneous tow-server queueing system with partial breakdown has been studied. The model studied in this article is more realistic for modeling queueing situations where the server may experienced many types of breakdowns which can be realized in manufacturing production systems. The system can be generalized by taking $C \geq 3$ customers.

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REFERENCES

1. M.O.Abou-El-Ata and A.L.Shawky, A simple approach for the slower server problem, *Commun Fac. Sci, uni. Anv series A1*, 88 (1999) 1-6.
2. G.Choudhury and L.Tadj, An $M/G/1$ queue with two phases of service subject to the server breakdown and delayed repair, *Applied Mathematical Modelling,* 33 (2009) 2699-2709.
3. J.H.A.Desmit, A numerical solution for the multi server queue with hyper exponential service times, *Oper. Res. Lett.*, 2 (1983) 217-224.
4. A.Federgruen and L.Green, Queueing systems with service interruptions, *Operations Research,* 34 (1986) 752-768.
5. J.C.Heffer, Steady state solution of the $M/E_k/C$(F1F0) queuing system, *CORSJ.,* 7 (1969) 16-30.
6. S.Karlin and J.Mc.Gregor, Many server queuing process with Poisson input and exponential service times, *Pacific J.Math.,* 8 (1958) 87-118.
7. D.G.Kendall, Stochastic process in the theory of queues, *Ann. Math. Stat.,* 24 (1953) 333-354.
8. J.Kiefer and J.Wolfowitz, On the theory of queues with many servers, *Trans Amer. Math.Ssoc.,* (1955) 781-18
9. B.Krishnamoorthi, On Poisson queue with two heterogeneous servers, *Oper. Res.*,11(3) (1963) 321-330.
10. B.K.Kumar and S.P.Madheswari, An $M/M/2$ queueing system with heterogeneous servers and multiple vacations, *Mathematical and Computer Modelling,* 41 (2005) 1415-1429.
11. G.Latouche and M.F.Neuts, Efficient algorithmic solutions to exponential tandem queues with blocking, *SIAM J. Algebraic Discrete Math.,* 1 (1980) 93-106.
12. W.Li, D.Shi and X.Chao, Reliability analysis of $M/G/1$ queueing systems with server breakdowns and vacations, *Journal of Applied Probability,* 34 (1997) 546-555.
13. W.Lin and P.Kumar, Optimal control of c queuing system with two heterogeneous servers, *Automatic Control, IEEE trans.,* 29(8) (1984) 696-703.
14. D.M.Lucantoni, A $GI/M/C$ queue with a different service rate for customers who need not wait an algorithmic solution, *Technical Rep. Univ. of Delware,* USA (1979).
15. K.C.Madan, W.Abu-Dayyeh and F.Taiyyan, A two server queue with Bernoulli schedules and a single vacation policy, *Applied Mathematics and Computation,* 145 (2003) 59-71.
16. I.L.Mitrany and B.Avi-Itzhak, A many-server queue with service interruptions, *Operations Research,* 16 (1968) 628-638.

17. O.Nakdimon and U.Yechiali, Polling systems with breakdowns and repairs, *European Journal of Operational Research,* 149 (2003) 588-613.
18. M.F.Neuts, Markov chains with applications in queueing theory which have a matrix-geometric invariant probability vector, *Adv. Appl. Probab.,* 10 (1978) 185-212.
19. M.F.Neuts, Matrix-Geometric solution in stochastic models, *Vol 2 of John Hopkins series in the Mathematical Sciences, Johns Hopkins University press, Baltimore, md, USA,* 1981.
20. M.F.Neuts and D.M.Lucantoni, A Markovian queue with N servers subject to breakdowns and repairs, *Management Science,* 25 (1979) 849-861.
21. G.Rani and V.Shanthi, M/M/2/K Loss and Delay Interdependent queueing model with controllable arrival rates, No passing and Feedback, *Annals of Pure and Applied Mathematics,* 15(1) (2017) 13-24.
22. M.Rubinovitch, The slow server problem; A queue with stalling, *Jr. Appl. Prob.,* 22 (1985) 879-892.
23. V.P.Singh, Two servers Markovian queues with balking, Heterogeneous vs Homogeneous servers, *Oper. Res.,* 18(1) (1970) 145-159.
24. V.P.Singh, Queue dependence-servers, *Jr. of Eng. Maths,* 7(2) (1973) 123-126.
25. Y.Tang, Single-server M/G/1 queueing system subject to breakdowns-some reliability and queueing problems, *Microelectronics Reliability,* 37 (1997) 315-321.
26. B.Vinod, Unreliable queueing systems, *Computers and Operations Research,* 12 (1985) 322-340.
27. K.H.Wang and Y.C.Chang, Cost analysis of a finite M/M/R queueing system with balking, reneging and server breakdowns. *Mathematical Methods of Operations Research,* 56 (2002) 169-180.
28. J.Wang, B.Liu and J.Li, Transient analysis of an M/G/1 retrial queue subject to disasters and server failures, *European Journal of Operational Research,* 189 (2008) 1118-1132.
29. K.Wang, T.Wang and W.Pearn, Optimal control of the N policy M/G/1 Queueing systems with server breakdowns and general startup times, *Applied Mathematical Modelling,* 31 (2007) 2199-2212.
30. P.Wartenhosrt, N parallel Queueing systems with server breakdowns and repair, *European Journal of Operational Research,* 82 (1995) 302-322.
31. D.Yue, J.Yu and W.Yue, A Markovian queue with two heterogeneous server and multiple vacations, *Journal of Industrial and Management Optimization,* 5 (2009) 453-465.