Annals of Pure and Applied Mathematics
Vol. 18, No. 1, 2018, 45-50
ISSN: 2279-087X (P), 2279-0888(online)
Published on 28 July 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v18n1a6

Annals of
Pure and Applied Mathematics

Semi Prime Filters in Join Semilattices

***Shiuly Akhter*[1] and *A.S.A. Noor*[2]**

[1]Departmentof Mathematics, Rajshahai University, Bangladesh
[2]Department of ETE, Prime University, Dhaka, Bangladesh
[2]Email: asanoor100@gmail.com
[1]Corresponding author. Email: shiuly_math_ru@yahoo.com

Received 2 June 2018; accepted 19 July 2018

Abstract. Theconcept of semi primefilters in a general lattice have been given by Ali et al. [2]. A filter F of a lattice L is called semi prime filter if for all $x, y, z \in L$, $x \vee y \in F$ and $x \vee z \in F$ imply $x \vee (y \wedge z) \in F$. In this paper we give several properties of semi prime filters in join semilattice and include some of their characterizations. Here we prove that a filter F is semi prime if and only if every maximal ideal of a directed below join semilattice S, disjoint with F is prime.

Keywords: Semi prime filter,Maximal ideal, Minimal prime filter, Annihilator filter.

***AMS Mathematics Subject Classification(2010)*:** 06A12, 06A99, 06B10

1. Introduction

Varlet [1] introduced the concept of 1-distributive lattices. Then many authors including [7] and [8] studied them for lattices and join semilattices. An ordered set $(S; \leq)$ is said to be a join semilattice if $\sup\{a, b\}$ exists for all $a, b \in S$. We write $a \vee b$ in place of $\sup\{a, b\}$. By [8], a join semilattice S with 1 is called 1-distributive if for all $a, b, c \in S$ with $a \vee b = 1 = a \vee c$ imply $a \vee d = 1$ for some $d \leq b, c$. We also know that a 1-distributive join semilattice S is directed below. A join semilattice S is called directed below if for all $a, b \in S$, ther exists $c \in S$ such that $c \leq a, b$. A non-empty subset F of a directed below join semilattice S is called up set if for $x \in F$ and $y \geq x (y \in S)$ imply $y \in F$. An up set F is called a filter if for $x, y \in F$, there exists $z \leq x, y$ such that $z \in F$.

A non-empty subset I of S is called a down set if $x \in I$ and $y \leq x (y \in S)$ imply $y \in I$. A down set I of S is called an ideal if for all $x, y \in I$, $x \vee y \in I$. A filter (up set) P is called a prime filter if $a \vee b \in P$ implies either $a \in P$or $b \in P$. An ideal J of S is called prime if $S - J$ is a prime filter.

An ideal I of S is called maximal ideal if $I \neq S$ and it is not contained by any other proper ideal of S. For $a \in S$, the filter $F = \{x \in S | x \geq a\}$ is called the principal filter generated by a. It is denoted by $[a)$. A prime up set (filter) is called a minimal prime up set (filter) if it does not contain any other prime up set (filter).

An ideal I of a lattice L is called a semi prime ideal if for all $x, y, z \in L$, $x \wedge y \in I$ and $x \wedge z \in I$ imply $x \wedge (y \vee z) \in I$. Thus, for a lattice L with 0, is called 0-distributive if and only if $(0]$ is a semi prime ideal. In a distributive lattice L, every ideal is a semi prime ideal. Moreover, every prime ideal is semi prime. For details of semi prime ideals and semi prime n-ideals in lattices see [3,4,6,9]. By [5], in a directed above meet semilattice S, an ideal J is called a semi prime ideal if for all $x, y, z \in S, x \wedge y \in J, x \wedge z \in J$ imply $x \wedge d \in J$ for some $d \geq y, z$.

A filter F of a lattice L is called semi prime filters if for all $x, y, z \in L, x \vee y \in F$ and $x \vee z \in F$ imply $x \vee (y \wedge z)$. Thus for a lattice L with 1, is called 1-distributive if and only if $[1)$ is a semi prime filter. In a distributive lattice L, every filter is a semi prime filter. Moreover, every prime filter is asemi prime.

In this paper, we extend the concept of semi prime filters for directed below join semilattice S and give several characterizations of semi prime filters. In a directed below join semilattice S, a filter F is called semi prime filter if for all $x, y, z \in S, x \vee y \in F$ and$x \vee z \in F$ imply $x \vee d \in F$ for some $d \leq y, z$. In a distributive semilattice, every filter is semi prime filter. Moreover, the semilattice itself is obviously a semi prime filter. Also, every prime filter of S is semi prime.

2. Main results

To obtain the main results of this paper we need to prove the following lemmas.

Lemma 1. Intersection of two prime (semi prime) filters of a directed below join semilattice S is a semi prime filter.

Proof: Let $x, y, z \in S$ and $F = P_1 \cap P_2$. Let $x \vee y \in F$and $x \vee z \in F$. Then $x \vee y \in P_1$, $x \vee z \in P_1$ and $x \vee y \in P_2$, $x \vee z \in P_2$. Since P_1 and P_2 are prime(semi prime) filters, so $x \vee d_1 \in P_1$ and $x \vee d_2 \in P_2$ for some $d_1, d_2 \leq y, z$. Choose $d = d_1 \vee d_2 \leq y, z$. Then $x \vee d \in F$ i.e. $x \vee d \in P_1 \cap P_2$ and so $P_1 \cap P_2$ is semi prime filter.

Corollary 2. Non empty intersection of all prime (semi prime) filters of a directed below join semilattice is a semi prime filter.

Following two lemmas are due to [8].

Lemma 3. A proper subset I of a join semilalttice S is a maximal ideal if and only if $S - I$ is a minimal prime up set (filter).

Lemma 4. Let I be a proper ideal of a join semilattice S with 1.Then there exists a maximal ideal containg I.

Lemma 5. Every ideal disjoint from a filter F is contained in a maximal ideal disjoint from F.

Proof: Let I be an ideal in a directed below joint semilattice S disjoint from F. Let J be set of all ideals containing I and disjoint from F. Then J is nonempty as $I \in J$. Let C be a chain in J and let $M = U(X : X \in C)$. We claim that M is an ideal. Let $x \in M$ and $y \leq x$.Then$x \in X$ for some $X \in C$. Hence $y \in X$ as X is an ideal. Thus, $y \in M$. Let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in C$. Since C is a chain, either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$, so $x, y \in Y$. Then $x \vee y \in Y$ and so $x \vee y \in M$. Hence M is an ideal Moreover, $M \cap F = \Phi$ and $M \supseteq I$. Thus M is a maximal element of C. Therefore, by Zorn's Lemma, J has a maximal element.

Lemma 6. Let F be a filter of a directed below semilattice$S$. An ideal I disjoint from F is a maximal ideal disjoint from F if and only if for all $a \notin I$, there exists $b \in I$ such that $a \vee b \in F$.

Proof: Let I be a maximal ideal and disjoint from F and let $a \notin I$.Also let $a \vee b \notin F$ for all $b \in I$. Consider $M = \{y \in S | y \leq a \vee b, b \in I\}$. Clearly, M is an ideal. For any $b \in I$, $b \leq a \vee b$ implies $b \in M$. Hence$M \supseteq I$. Also $M \cap F = \Phi$. For if not, let $x \in M \cap F$ which implies $x \in F$ and $x \leq a \vee b$ for some $b \in I$. Hence $a \vee b \in F$ which is a contradiction. Thus $M \cap F = \Phi$. Now $M \supset I$ because $a \notin I$ but $a \in M$. This contradicts the maximality of I. Hence there exists$b \in I$ such that $a \vee b \in F$.

Conversely, if I is not maximal ideal disjoint from F, then there exists an ideal J containing I disjoint with F. Let $a \in J - I$ by the given condition there exists $b \in I$ such that $a \vee b \in F$. Hence $a, b \in J$ implies $a \vee b \in F \cap J$ which is a contradiction. Therefore, I must be a maximal ideal disjoint from F.

Theorem 7. A join semilattice S with at least one proper semi prime filter is directed below.

Proof: Let $a, b \in S$ and F be a semi prime filter of S. Then for any $x \in F$, $x \vee a \in F$ and $x \vee b \in F$. Since F is semi prime, so there exists $d \in S$ with $d \leq a, b$ such that $x \vee d \in F$. Hence S is directed below.

Let L be a lattice with 0. For $A \subseteq L$, we define $A^{\perp} = \{x \in L: x \wedge a = 0$ for all $a \in A\}$.

Let S be a join semilattice with 1. For a non-empty subset A of S, we define$A^{\perp^d} = \{x \in S | x \vee a = 1$ for all $a \in A\}$. This is clearly an up set but we can not prove that this is a filter even in a distributive join semilattice. If L is a lattice with 1, then it is well known that L is 1-distributive if and only if $D(L)$, the lattice of all filters of L is 0-distributive. Unfortunately, we can not prove or disprove that when S is a 1-distributive join semilattice, then $D(S)$ is 0-distributive. But if $D(S)$ is 0-distributive, then it is easy to prove that S is 1-distributive.

Also we define $A^1 = \{x \in S | x \vee a = 1$ for some $a \in S\}$. This is obviously an up set. Moreover, $A \subseteq B$implies $A^1 \subseteq B^1$. For any $a \in S$, it is easy to check that $(a)^{\perp^d} = (a)^1 = (a]^1$. Since in a 1-distributive join semilattice S, for each $a \in S$, $(a)^{\perp^d}$ is a filter, so we prefer to denote it by $[a)^{*d}$. Let $A \subseteq S$ and P be a filter of L. We define $A^{\perp^{d_P}} = \{x \in S | x \vee a \in P$ for all $a \in A\}$. This is clearly an up set containing P. In presence of distributivity, this is a filter. $A^{\perp^{d_P}}$ is called a dual annihilator of A relative to P, we denote $F_P(S)$, by the set of all filters containing P. Of course $F_P(S)$ is a bounded lattice with P and S as the smallest and the largest elements. If $A \in F_P(S)$ and $A^{\perp^{d_P}}$ is a filter, then $A^{\perp^{d_P}}$ is called an annihilator filter and it is the dual pseudocomplement of A in $F_P(S)$.

Theorem 8. Let S be a directed below join semilattice with 1 and P be a filter of S. Then the following conditions are equivalent :

(i) P is semi prime

(ii) For every $a \in S$,$\{a\}^{\perp^{d_P}} = \{x \in S | x \vee a \in P\}$ is a semi prime filter containing P.

(iii) $A^{\perp^{d_P}} = \{x \in P | x \vee a \in P$ for all $a \in A\}$ is a semi prime filter containing P.

(iv) Every maximal ideal disjoint from P is prime

Proof: (i) ⇒ (ii). Clearly $\{a\}^{\perp^{d_P}}$ is an up set containing P. Now let $x, y \in \{a\}^{\perp^{d_P}}$. Then $x \vee a \in P$, $y \vee a \in P$. Since P is semi prime, so $a \vee d \in P$ for some $d \leq x, y$. Thus $d \in \{a\}^{\perp^{d_P}}$. This implies $\{a\}^{\perp^{d_P}}$ is a filter containing P. Again let $x \vee y \in \{a\}^{\perp^{d_P}}$ and $x \vee z \in \{a\}^{\perp^{d_P}}$. Then $x \vee y \vee a \in P$ and $x \vee z \vee a \in P$. Hence $(x \vee a) \vee y \in P$ and $(x \vee a) \vee z \in P$. Then $(x \vee a) \vee d \in P$ for some $d \leq y, z$, as P is semi prime. This implies $x \vee d \in \{a\}^{\perp^{d_P}}$ and so $\{a\}^{\perp^{d_P}}$ is a semi prime filter containing P.

(ii) ⇒ (i). Suppose (ii) holds. Let $x \vee y \in P$ and $x \vee z \in P$. Then $y, z \in \{x\}^{\perp^{d_P}}$. Since by (ii),$\{x\}^{\perp^{d_P}}$ is a filter, so there exists $d \leq y, z$ such that $d \in \{x\}^{\perp^{d_P}}$. Thus $x \vee d \in P$ and so P is semi prime. (ii) ⇒ (iii). This is trivial by Lemma 1 as $A^{\perp^{d_P}} = \cap\left(\{a\}^{\perp^{d_P}}; a \in A\right)$.

(i) ⇒ (iv). Suppose J is a maximal ideal disjoint from P. Suppose $f, g \in S - J$. Then $f, g \notin J$. By Lemma 6, there exist $a, b \in J$ such that $a \vee f \in P$, $b \vee g \in P$. Here $S - J$ is a minimal prime up set containing P. Hence $a \vee b \vee f \in P$ and $a \vee b \vee g \in P$. Since P is semi prime, so there exists $e \leq f, g$ such that $a \vee b \vee e \in P \subseteq S - J$. But $a \vee b \in J$ and so $e \in S - J$ as it is prime. Here $S - J$ is a prime filter. Hence J is a prime ideal. (iv) ⇒ (i). Let (iv) holds. Suppose $a, b, c \in S$ with $a \vee b \in P, a \vee c \in P$. Suppose $a \vee d \notin P$ for all $d \leq b, c$. Consider $J = \{y \in S | y \leq a \vee d; d \leq b, c\}$. Then J is an ideal disjoint from P. By Lemma 5, there is a maximal ideal $M \supseteq J$ and disjoint from P. By (iv) M is prime. Thus $S - M$ is a prime filter containing P.

Now $a \vee b, a \vee c \in S - M$. Since $S - M$ is a prime filter , so either $a \in S - M$ or $b, c \in S - M$. In any case, $a \vee d \in S - M$ for some $d \leq b, c$. This gives a contradiction as $a \vee d \in M$ for all $d \leq b, c$. Hence $a \vee d \in P$ for some $d \leq b, c$. Therefore, P is semi prime.

Corollary 9. In a join semilattice S, every ideal disjoint to a semi prime filter P is contained in a prime ideal.

Theorem 10. If P is a semi prime filter of directed below join semilattice S and $P \subset A = \cap\{P_\lambda | P_\lambda$ is a filter containing $P\}$. Then $A^{\perp^{d_P}} = \{x \in S | \{x\}^{\perp^{d_P}} \neq P\}$.

Proof: Let $x \in A^{\perp^{d_P}}$. Then $x \vee a \in P$ for all $a \in A$. So $a \in \{x\}^{\perp^{d_P}}$ for all $a \in A$. Then $A \subseteq \{x\}^{\perp^{d_P}}$ and so $\{x\}^{\perp^{d_P}} \neq P$.

Conversely, let $x \in S$ such that $\{x\}^{\perp^{d_P}} \neq P$. Since P is semi prime, so $\{x\}^{\perp^{d_P}}$ is a filter containing P. Then $A \subseteq \{x\}^{\perp^{d_P}}$ and so $A^{\perp^{d_P}} \supseteq \{x\}^{\perp^{d_P}\perp^{d_P}}$. This implies $x \in A^{\perp^{d_P}}$ which completes the proof.

Theorem 11. Let S be a directed below join semilattice and F be a filter. Then the following conditions are equivalent:

(i)F is semi prime.

(ii) Every maximal ideal of S disjoint with F is prime.

(iii) Every minimal prime up set containing F is a minimal prime filter containing F.

(iv) Every ideal disjoint with F is disjoint from a minimal prime filter containing F.

Proof. (i) ⇔ (ii) follows from Theorem 8.

(ii) ⇒ (iii). Let A be a minimal prime up set containg F. Then $S - A$ is a maximal ideal disjoint with F. Then by (ii), $S - A$ is a prime ideal and so A is a minimal prime filter.

(iii) $\Rightarrow$ (ii). Le M be a maximal ideal disjoint with F. Then $S-M$ is a minimal prime up set containing F. Then by (iii), $S-M$ is a minimal prime filter and so M is a prime ideal.
(i) $\Rightarrow$ (iv). Let I be an ideal of S disjoint from F. Then there exists a maximal ideal $J \supseteq I$ disjoint to F. By Theorem8, J is a prime ideal and so $S-J$ is a minimal prime filter containing F and disjoint from I.
(iv) $\Rightarrow$ (ii). Let J be maximal ideal disjoint from F. Then by (iv), there exists a minimal prime filter P containing F and disjoint from J. Then $S-P$ is a maximal prime ideal of S containing J and disjoint from F. By maximality of $J, S-P$ must be equal to J. Hence J is prime.

Theorem 12. Let S be a directed below join semilattice with 1 and P be a filter of $S. P$ is semi prime if and only if for all ideals$I$ disjoint to $\{x\}^{\perp^{d}P}$ there is a prime ideal containing I disjoint to $\{x\}^{\perp^{d}P}$.

Proof. Suppose P is semi prime. Then by Theorem 8, $\{x\}^{\perp^{d}P}$ is semi prime. Let I be an ideal disjoint to $\{x\}^{\perp^{d}P}$. Using Zorn's Lemma we can easily find a maximal ideal M containing I and disjoint to $\{x\}^{\perp^{d}P}$. We claim that $x \in M$. If not, then $M \vee (x] \supset M$. By maximality of M, $(M \vee (x]) \cap \{x\}^{\perp^{d}P} \neq \Phi$. If $t \in (M \vee (x]) \cap \{x\}^{\perp^{d}P}$, then $t \leq m \vee x$ for some $m \in M$ and $t \vee x \in P$. This implies $m \vee x \in P$ and so $m \in \{x\}^{\perp^{d}P}$ gives a contradiction. Hence $x \in M$. Now let $z \notin M$. Then $(M \vee (z]) \cap \{x\}^{\perp^{d}P} \neq \Phi$. Suppose $y \in (M \vee (z]) \cap \{x\}^{\perp^{d}P}$ then $y \leq m_1 \vee z$ and $y \vee x \in P$ for some $m_1 \in M$. This implies $m_1 \vee x \vee z \in P$ and $m_1 \vee z \in \{x\}^{\perp^{d}P}$. Hence by Lemma 6, M is a maximal ideal disjoint to $\{x\}^{\perp^{d}P}$. Therefore, by theorem 8, M is prime.
Conversely, let $x \vee y \in P, x \vee z \in P$. If $x \vee d \notin P$ for all $d \leq y, z$ then $d \notin \{x\}^{\perp^{d}P}$. Hence$(d] \cap \{x\}^{\perp^{d}P} = \Phi$. So there exists a prime ideal M containing $(d]$ and disjoint from $\{x\}^{\perp^{d}P}$. As $y, z \in \{x\}^{\perp^{d}P}$, so $y, z \notin M$. Thus $d \notin M$ for some $d \leq y, z$ as M is prime. This gives a contradiction. Hence $x \vee d \in P$ for all $d \leq y, z$ and so P is semi prime.

Corollary 13. A directed below join semilattice S with 1 is 1-distributive if and only if every prime up set contains a minimal prime filter.
Proof. Let P be a prime up set of S. Then $P \neq S$. So there exists $x \in S$ such that $x \notin P$.If $t \in \{x\}^{\perp^{d}}$, then $t \vee x = 1 \in P$. This implies$t \in P$, as P is prime.
Hence $\{x\}^{\perp^{d}} \cap (S-P) = \Phi$, where $S-P$ is an ideal of S. Suppose S is 1-distributive (i.e. [1) is semi prime). Then by Theorem 12, there is prime ideal J containing $S-P$ and disjoint to $\{x\}^{\perp^{d}}$. This implies that $S-J$ is a minimal prime filter contained in P. Proof of the converse is trivial from the proof of Theorem 12.
We conclude the paper with the following characterization of semi prime filters.

Theorem 14. Let P be a semi prime filter of a directed below join semilattice S and $x \in S$. Then a prime filter Q containing $\{x\}^{\perp^{d}P}$ is a minimal prime filter containing $\{x\}^{\perp^{d}P}$ if and only if for $q_1 \in Q$, there exists $q_2 \in S-Q$ such that $q_1 \vee q_2 \in \{x\}^{\perp^{d}P}$.
Proof. Let Q be a prime filter containing $\{x\}^{\perp^{d}P}$ such that the given condition holds. Let R be a prime filter containing$\{x\}^{\perp^{d}P}$ such that $R \subseteq Q$. Let $q_1 \in Q$.Then there is $q_2 \in S -$

Q suchthat $q_1 \vee q_2 \in \{x\}^{\perp^{d_P}}$. Hence $q_1 \vee q_2 \in R$. Since R is prime and $q_2 \notin R$, so $q_1 \in R$. Thus $Q \subseteq R$ and so $R = Q$. Therefore, Q must be a minimal prime filter containing $\{x\}^{\perp^{d_P}}$.

Conversely, let Q be a minimal prime filter containing $\{x\}^{\perp^{d_P}}$. Let $q_1 \in Q$. Suppose for all $q_2 \in S - Q, q_1 \vee q_2 \notin \{x\}^{\perp^{d_P}}$. Let $I = (S - Q) \vee (q_1]$. We claim that $\{x\}^{\perp^{d_P}} \cap I = \Phi$. If not, let $y \in \{x\}^{\perp^{d_P}} \cap I$. Then $y \in \{x\}^{\perp^{d_P}}$ and $y \leq q_1 \vee q_2$. Thus $q_1 \vee q_2 \in \{x\}^{\perp^{d_P}}$, which is a contradiction to the assumption. Then by Theorem 12, there exists a maximal prime ideal $M \supseteq I$ and disjoint to $\{x\}^{\perp^{d_P}}$. Let $J = S - M$. Then J is a prime filter containing $\{x\}^{\perp^{d_P}}$. Now $J \cap I = \Phi$. This implies $J \cap (S - Q) = \Phi$ and so $J \subseteq Q$. Also $J \neq Q$, because $q_1 \in I$ implies $q_1 \notin J$ but $q_1 \in Q$. Hence J is a prime filter containing $\{x\}^{\perp^{d_P}}$ which is properly contained in Q. This gives a contradiction to the minimal property of Q. Therefore, the given condition holds. That is, for $q_1 \in Q$, there exists $q_2 \in S - Q$ such that $q_1 \vee q_2 \in \{x\}^{\perp^{d_P}}$.

3. Conclusion

In this paper, we extend the concept of semi prime filters in directed below join semilattices and include several nice characterizations of semi prime filters. We also prove some interesting results on semi prime filters in directed below join semilattices. Here we prove that, a filter F is semi prime if and only if every maximal ideal of a directed below join semilattice, disjoint with$F$ is prime.

Acknowledgements.The authors are grateful and thankful to the reviewers for their valuable comments and suggestions for the improvement of the paper.

REFERENCES

1. J.C.Varlet, A generalization of the notion of pseudo-complementedness, *Bull. Soc. Sci. Liege*, 37 (1968) 149-158.
2. M.A.Ali,M.Begum and A.S.A.Noor, On semi prime filters in lattices, *Annals of Pure and Applied Mathematics*, 12(2) (2016) 129-136.
3. M.A.Ali, R.M.H.Rahman and A.S.A.Noor, Some properties of semi prime ideals in lattices, *Annals of Pure and Applied Mathematics*, 1(2) (2012) 176-185.
4. M.A.Ali, R.M.H.Rahman and A.S.A.Noor, On semi prime n-ideals in lattices, *Annals of Pure and Applied Mathematics*, 2(1) (2012) 10-17.
5. M.Begum and A.S.A.Noor, Semi prime ideals in meet semi lattices, *Annals of Pure and Applied Mathematics*, 1(2) (2012) 176-185.
6. R.M.H.Rahman, M.A.Ali and A.S.A.Noor, On semi prime ideals of a lattice, *J. Mech. Cont. & Math. Sci.*, 7(2) (2013) 1094-1102.
7. R.Sultana, M.A.Ali and A.S.A.Noor, Some properties of 0-distributive and 1-distributive Lattices, *Annals of Pure and Applied Mathematics*, 1(2) (2012) 168-175.
8. S.Akhter and A.S.A.Noor, 1-distributive join semilattice, *J. Mech. Cont. & Math. Sci.*, 7(2) (2013) 1067-1076.
9. Y.Rav, Semi prime ideals in general lattices, *Journal of pure and applied algebra*, 56 (1989) 105-118.