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On Solutions to the Diophantine Equation $x^3 - y^2 = z^2$

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Abstract. In this note, we investigate solutions to the title equation. We exhibit a solution in which $x \neq y$ and x, y are primes. When $x = y$, it is established that the equation has infinitely many solutions: (i) For x prime. (ii) For x composite of odd and even values. Several such solutions are demonstrated.

Keywords: Diophantine equations

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are for example [1, 3, 4, 6]. The title equation stems from $p^x \pm q^y = z^2$.

Whereas in most articles, the values x, y are investigated for the solutions of the equation, in this paper these values are fixed positive integers. In [3] we have considered the equation $p^3 + q^2 = z^2$ for all primes $p \geq 2$ and $q > 1$ prime or composite. In the equation $x^3 - y^2 = z^2$, the values x, y, z are positive integers, and we are interested in how many solutions exist when x, y are primes and also composites.

2. Solutions of the equation $x^3 - y^2 = z^2$

In this section we establish the following result.

Theorem 2.1. The equation $x^3 - y^2 = z^2$ in positive integers x, y, z has:

(a) At least one solution when $x \neq y$ and x, y are primes.
(b) Infinitely many solutions when $x = y$, and x is prime.
(c) Infinitely many solutions when $x = y$, and x is composite.

Proof: (a) In $x^3 - y^2 = z^2$ suppose that x, y are distinct primes. When $x = 5$ and $y = 11$ we have

Solution 1. $5^3 - 11^2 = 2^2$.

Thus, $x^3 - y^2 = z^2$ has at least one solution when $x < y$ are primes, and case **(a)** is complete.

(b) In $x^3 - y^2 = z^2$ suppose that $x = y$ and x is prime. When $x = y$, then

$$x^3 - y^2 = x^2(x - 1) = z^2, \qquad z \text{ is even} \qquad (1)$$

implying in (1) that $x - 1$ must be a square. Denote $x - 1 = T^2$, and $x = T^2 + 1$ where $T = 1, 2, \dots, n, \dots$. Certainly, x is prime only when T assumes even values. Moreover, when x is an odd prime p, then p is of the form $x = p = 4N + 1$.

The equation $x = T^2 + 1$ generates infinitely many composites, and also infinitely many primes all of which besides $p = 2$ are of the form $p = 4N + 1$. The value $T = 2$ yields $x = p = 4N + 1 = 5$. The prime $p = 5$ is a unique prime occurring once in the line of all primes. Therefore, we consider the primes whose last digit is equal to 1, 3, 7 and 9. If the last digit of $x = p$ ends in 3 or in 9, then the square $x - 1 = T^2$ ends in the digit 2 or respectively in 8. Since no even square can end in the digit 2 and in the digit 8, it follows that $x = p$ must end in the digit 1 or in the digit 7. All of the above conditions may be seen in the following Table 1 when the first ten values of T are demonstrated.

Table 1:

T	***x* - prime**	***x*- composite**		***T***	***x* - prime**	***x*- composite**
1	2			6	37	
2	5			7		50
3		10		8		65
4	17			9		82
5		26		10	101	

We now exhibit the five solutions of $x^3 - y^2 = z^2$ which relate to the five primes p contained in Table 1.

Solution 2. $2^3 - 2^2 = 2^2$,
Solution 3. $5^3 - 5^2 = 10^2 = (2\cdot5)^2$,
Solution 4. $17^3 - 17^2 = 68^2 = (2^2\cdot17)^2$,
Solution 5. $37^3 - 37^2 = 222^2 = (2\cdot3\cdot37)^2$,
Solution 6. $101^3 - 101^2 = 1010^2 = (2\cdot5\cdot101)^2$.

There are infinitely many primes $p = 4N + 1$ whose last digit is equal to 1, and as well with last digit equal to 7. Each such set has a subset of infinite primes $p = T^2 + 1$ where T is even. Hence, there are infinitely many solutions of $x^3 - y^2 = z^2$ where $x = y$ and x is a prime $p = T^2 + 1 = 4N + 1$.

This concludes case **(b)**.

(c) In $x^3 - y^2 = z^2$ suppose that $x = y$ and x is composite.

Since $x = y$, therefore x, y are both even or both odd. Table 1 yields the smallest odd composite $c = 65$ ($T = 8$). The next odd value $c = T^2 + 1$ is $c = 145$ ($T = 12$). The two solutions of $x^3 - y^2 = z^2$ are then:

Solution 7. $65^3 - 65^2 = 520^2 = (2^3\cdot5\cdot13)^2$,
Solution 8. $145^3 - 145^2 = 1740^2 = (2^2\cdot3\cdot5\cdot29)^2$.

When c is an even composite, Table 1 provides the first four such values, i.e., T = 3, 5, 7, 9, and respectively $x = c$ = 10, 26, 50, 82. The first two solutions of which are:
Solution 9. $10^3 - 10^2 = 30^2 = (2 \cdot 3 \cdot 5 \cdot)^2$,
Solution 10. $26^3 - 26^2 = 130^2 = (2 \cdot 5 \cdot 13)^2$.

For each and every value $T > 1$, there exists a value $x = T^2 + 1$. It therefore follows that there exist infinitely many composites $x = c$ where x is either odd or x is even. Case **(c)** is complete.

This concludes the proof of Theorem 2.1. □

3. Conclusion

We have shown that the equation $x^3 - y^2 = z^2$ has infinitely many solutions in which $x = y$, when x is prime, and x is composite odd or even. When $x \neq y$, and x, y are primes, we have demonstrated the solution $5^3 - 11^2 = 2^2$ (**Solution 1**). This is the only solution of this case known to us thus far. We may now raise the following two questions.

Question 1. Do there exist solutions of $x^3 - y^2 = z^2$ in which x, y are distinct primes other than 5 and 11 ?

Question 2. Do there exist solutions of $x^3 - y^2 = z^2$ in which x, y are distinct composites ?

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