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On the Non-Linear Diophantine Equation $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$

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***Abstract*.** In this paper, we consider the non-linear Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$, where *x, y* and *z* are non-negative integers. It has been shown that these non-linear Diophantine equations have no solution.

Keywords: Diophantine Equations, Catalan's Conjecture, Exponential Equations.

AMS Mathematics Subject Classification(2010): 11D61

1. Introduction

If a Diophantine equation has as an additional variable or variables occurring as exponents, it is an exponential Diophantine equation like as the equation of the Fermat- Catalan conjecture and Beal's conjecture, $a^m + b^n = c^k$ with inequality restrictions on the exponents. A general theory for such equations is not available; particular cases such as Catalan's conjecture have been tackled. In 1884, Catalan [5] conjectured that the four tuple $(a, b, x, y) = (3, 2, 2, 3)$ is the unique solution of the Diophantine equation $a^x - b^y = 1$, where *a, b, x* and *y* are non-negative integers with min $\{a, b, x, y\}>1$. In 2004, Mihailescu [6] proved the Catalan's conjecture and Corollary that $(p, x, y) = (2, 3, 3)$ is the unique solution of the Diophantine equation $1+ p^x = z^2$, where *p* is prime and min$\{p, x, y ,\}>1$. In 2011, A. Suvarnamani [7] showed that the Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no solution, where *x, y* and *z* are non-negative integers. In 2012, Sroysang [8] proved that the Diophantine equation $8^x + 19^y = z^2$ has the unique solution $(x, y, z) = (1, 0, 3)$, where *x, y* and *z* are non-negative integers. In 2013, Sroysang [9] proved that the Diophantine equation $7^x + 8^y = z^2$, where *x, y* and *z* are non-negative integers, has the unique solution $(x, y, z) = (0, 1, 3)$. In 2014, Sroysang [10] proved that the Diophantine equation $4^x + 10^y = z^2$ has no solution, where *x, y* and *z* are non-negative integers. In 2014, Sroysang [11] proved that the Diophantine equations $7^x + 19^y = z^2$ and $7^x + 91^y = z^2$, have no solution, where *x, y* and *z* are non-negative integers. In 2017, Acu [1] showed that the Diophantine equation $2^x + 5^y = z^2$, where *x, y* and *z* are non-negative integers, has only two solutions $(x, y, z) = (3, 0, 3)$ and $(2, 1, 3)$. In 2018, Burshtein [3] written a note on the Diophantine equation $2^a + 7^b = c^2$, where a and b are odd integers. In 2018, Burshtein [2] discussed on an open problem of Chotchaisthit, on the Diophantine

equation $2^x + p^y = z^2$, where p are particular prime and y=1. In 2018, Burshtein [4] also discussed on the Diophantine equation $2^x + p^y = z^2$, where p are prime.
In this paper we consider some particular exponential Diophantine equations

$$61^x + 67^y = z^2 \quad (1)$$

and

$$67^x + 73^y = z^2 \quad (2)$$

where x, y , and z are non-negative integers. We will use the Catalan's conjecture and congruency theory to solve these non-linear Diophantine equations.

2. Preliminaries

Proposition 2.1. (a, b, x, y) = (3, 2, 2, 3) is the unique solution of the Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers with min {a, b, x, y } >1.
Proof: See in [6].

Lemma 2.1. The Diophantine equation $1+ 67^y = z^2$ has no solution, where y and z are non-negative integers.
Proof: Let y and z are non-negative integers. Then we consider three cases.
Case I. If y = 0. Then $z^2 = 2$, which is not possible.
Case II. If y = 1. Then $z^2 = 68$, also not possible.
Case III. If y > 1. Then $z^2 = 1+67^y > 68$.
This implies z > 8. Here min { y, z} > 1, by Proposition, no solution.

Lemma 2.2. The Diophantine equation $1+ 73^y = z^2$ has no solution, where y and z are non-negative integers.
Proof: Let y and z are non-negative integers. Then we consider three cases.
Case I. If y = 0. Then $z^2 = 2$, which is not possible.
Case II. If y = 1. Then $z^2 = 74$, also not possible.
Case III. If y > 1. Then $z^2 = 1+73^y > 74$.
This implies z > 8. Here min { y, z} > 1, by Proposition, no solution.

Lemma 2.3. The Diophantine equation $61^x + 1 = z^2$ has no solution, where x and z are non-negative integers.
Proof: Let x and z are non-negative integers. Then we consider three cases.
Case I. If x = 0. Then $z^2 = 2$, which is not possible.
Case II. If x = 1. Then $z^2 = 62$, also not possible.
Case III. If x >1. Then $z^2 = 61^x + 1 > 62$.
This implies z > 7. Here min { x, z} > 1, by Proposition, no solution.

Lemma 2.4. The Diophantine equation $67^x + 1 = z^2$ has no solution, where x and z are non-negative integers.

Proof: Let x and z are non-negative integers. Then we consider three cases.
Case I. If x = 0. Then $z^2 = 2$, which is not possible.
Case II. If x = 1. Then $z^2 = 68$, also not possible.
Case III. If x >1. Then $z^2 = 67^x + 1 > 68$.
This implies z > 8. Here min { x, z} > 1, by Proposition, no solution.

3. Main theorem

Theorem 3.1. The non-linear Diophantine equation $61^x + 67^y = z^2$ has no solution, where x, y , and z are non-negative integers.

Proof: Let *x, y* , and *z* are non-negative integers. Then there are three cases.
Case I. If x=0, then by Lemma 2.1, there is no non-negative integer solution.
Case II. If $x\geq 1$and y=0, then by Lemma 2.3, also has no non-negative integer solution.
Case II. If $x\geq 1$and $y\geq 1$, then 61^x and 67^y both are odd. Thus z^2 is even, then $z^2 \equiv 0 \pmod 3$ or $z^2 \equiv 1 \pmod 3$. Since $61 \equiv 1 \pmod 3$ and $67 \equiv 1 \pmod 3$ then $61^x \equiv 1 \pmod 3$ and $67^y \equiv 1 \pmod 3$. Therefore $z^2 = 61^x + 67^y \equiv 2 \pmod 3$, which is a contradiction.

Corollary 3.1.1. The non-linear Diophantine equation $61^x + 67^y = k^{2t}$ has no solution, where *x, y* , and *z* are non-negative integers, *k* and *t* are positive integer.
Proof: Suppose the non-linear Diophantine equation $61^x + 67^y = k^{2t}$, where *x, y* , and *z* are non-negative integers, *k* and *t* are positive integer. Let $k^t = z$, then Diophantine equation becomes $61^x + 67^y = z^2$, which has no solution by Theorem 3.1.

Corollary 3.1.2. The non-linear Diophantine equation $61^x + 67^y = k^{2t+4}$ has no solution, where *x, y* , and *z* are non-negative integers, *k* and *t* are positive integer.
Proof: Let $k^{t+2} = z$, then Diophantine equation becomes $61^x + 67^y = z^2$, which has no solution by Theorem 3.1.

Theorem 3.2. The non-linear Diophantine equation $67^x + 73^y = z^2$ has no solution, where *x, y* , and z are non-negative integers.
Proof: Let *x, y* , and z are non-negative integers. Then there are three cases.
Case I. If x=0, then by Lemma 2.2, there is no non-negative integer solution.
Case II. If $x\geq 1$ and y=0, then by Lemma 2.4, also has no non-negative integer solution.
Case II. If $x\geq 1$ and $y\geq 1$, then 67^x and 73^y both are odd. Thus z^2 is even, then $z^2 \equiv 0 \pmod 3$ or $z^2 \equiv 1 \pmod 3$. Since $67 \equiv 1 \pmod 3$ and $73 \equiv 1 \pmod 3$ then $67^x \equiv 1 \pmod 3$ and $73^y \equiv 1 \pmod 3$. Therefore $z^2 = 67^x + 73^y \equiv 2 \pmod 3$, which is a contradiction.

Corollary 3.2.1. The non-linear Diophantine equation $67^x + 73^y = k^{2t}$ has no solution, where *x, y* , and z are non-negative integers, k and t are positive integer.
Proof: Suppose the non-linear Diophantine equation $67^x + 73^y = k^{2t}$, where *x, y* , and z are non-negative integers, k and t are positive integer. Let $k^t = z$, then Diophantine equation becomes $61^x + 67^y = z^2$, which has no solution by Theorem 3.2.

Corollary 3.2.2. The non-linear Diophantine equation $67^x + 73^y = k^{2t+4}$ has no solution, where *x, y* , and z are non-negative integers, k and t are positive integer.
Proof: Let $k^{t+2} = z$, then Diophantine equation becomes $67^x + 73^y = z^2$, which has no solution by Theorem 3.2.

4. Conclusion

In this paper, we discussed the non linear Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$ and find that these Diophantine equations have no solution for any non negative integers *x, y* and z .

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